REDUCTION OF BURMESTER THEORY
TO DESIGN PRACTICE

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REDUCTION OF BURMESTER THEORY

TO DESIGN PRACTICE

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SUMMARY

The Burmester problem has long been one of the major analytical approaches in the path and position synthesis of planar mechanisms. Recently this approach has been extended to three dimensions by the works of Roth and Williams and, later, by Sandor.

There have been limited attempts to provide the designer with an approach to synthesis of a planar linkage without actually solving the Burmester problem. These range from the giant atlas of planar four-bar mechanisms of Hrones and Nelson for coupler curve path synthesis to the design graphs of Tesar and others for the special case of the straight-line path generating mechanisms. While several methods for spatial mechanism synthesis have been presented, very few solutions have been produced and no design approaches have been developed which permit a non-specialist in kinematic synthesis to utilize such devices.

The development of a system of design graphs for three dimensional path synthesis was attempted but was not accomplished. This thesis presents a discussion on the problems of three dimensional design graphs. This thesis also presents several mechanisms which have been synthesized for a particular purpose and a compendium of spatial solutions which can be synthesized into linkages. These results do

provide a means for the non-specialist to synthesize a spatial linkage.

CHAPTER I

INTRODUCTION

Planar mechanism synthesis is usually concerned with obtaining one of three objectives. These are function generation where the positions of the input and output links are related by some mathematical function, path generation where a point on the coupler link of a mechanism is required to follow a particular path, and position synthesis where the coupler link is required to assume a sequence of positions as the input link is moved.

Path generation and position synthesis are closely related because paths are approximated by a sequence of points which are usually called accuracy points. The difference then is that a path generator is a position synthesizer that has only its positions specified along a path and has no specifications for angular displacements.

These three objectives are usually approached by one of two methods. The first of these methods involves solving for the unknown physical parameters of the linkage in the geometric equations which describe its positions. The second of these methods involves finding points in the coupler link whose successive positions lie on curves which can be mechanized. This second method derives from Burmester theory and has been used mainly in the design of path generators and position synthesizers.

Function generators can be designed for a variety of purposes such as mechanical linearizers, radio tuning mechanisms, constant angular-velocity-ratio couplings and linkages to replace cams [1]. Hartenburg and Denavit [2] show the design of planar function generators with three, four, and five accuracy points while Freudenstein [3] further provides a compendium of planar function generators whose error between the accuracy points has been minimized through computer optimization.

The planar Burmester problem was first solved graphically by Burmester [4] while important contributions were made to the theory by Mueller [5]. Modern development originated with analytical studies using complex numbers contributed by Freudenstein, Sandor, and Primrose [6,7,8,9]. Bottema treated the problem using the algebra of real numbers [10]. The special case where the specified positions are infinitesimally separated was treated by Mueller. More recently, Tesar and Eschenbach treated the case of a combination of finitely and infinitesimally separated positions [11]. Tesar has been responsible for much work in the area of single position design relating to straightline mechanisms and, with Vidosic and Wolford, has published a series of design graphs to use in the design of four-bar, straight-line generating linkages [12,13,14]. The most elementary compendium of planar solutions for occasional designers of path generators is the well known Hrones and Nelson atlas [15]. Brown and Mabie show applications of design curves to a variety of planar problems [16]. Freudenstein and Sandor have provided a digital computer program for the synthesis of path generating mechanisms [19].

Spatial linkage synthesis is similar to planar synthesis in the objectives and in the general methods of solution. However, three dimensional synthesis involves more types of joints and linkages than in planar synthesis which usually standardizes on the four-bar linkage with revolute joints. Also, in three dimensional synthesis, one linkage type is often best used for one type of synthesis while another type of linkage is best suited for another synthesis problem. Function generators in three dimensions have generally been limited to the spherical four bar, R-R-R-R, and the spatial four-bar, R-S-S-R (R = revolute joint, S = spherical joint). Hartenburg and Denavit [2] show the design of R-R-R-R and R-S-S-R function generators with three and six accuracy points, respectively, while Suh [17] shows the design of two RSSR function generators with four and six accuracy points. Suh and Radcliffe [18] show a method for the design of a spherical function generator but also include the design of a path generator for the special case of spherical path. Johnson [20] has shown a graphical-analytical method for the approximate synthesis of three dimensional path generation mechanisms where the path is limited to lie on the surface of a sphere.

Several investigators have provided means for finding surfaces in three dimensions which can be mechanized. Roth, in a series of significant papers [21,22,23] has presented methods for the treatment of mixed finitely and infinitesimally separated positions (with Chen) and for the solution of points which lie on the surface of a sphere, a circle, a plane, a cylinder, and a straight line. In a slightly earlier work, Williams [24] solved for these surfaces by using the algebra of real numbers and provided an analysis of the maximum number of arbitrary positions and related variables. This solution limited the movement between points by allowing three displacements but only one rotation, however, the input positions are defined only by their displacements relative to the stationary frame and do not have to be converted into screw displacements. Sandor [25] was able to duplicate the solutions of Roth and Williams for the case of the circle and recommended comparable linkages to that of Williams for position synthesis.

While many approaches to three dimensional path generation and position synthesis have been presented in the literature, very few solutions have been presented. The work of Freudenstein [1,19], Hrones and Nelson [15], and Tesar [12,13,14] represent some of the efforts which have been made on planar synthesis to free the designer from some of the computations and from the need to have a complete understanding of the theory. These papers represent the main approaches that a kinematician could use to make synthesis more available to the designer. These include: (1) the presentation of a specific linkage that will do a specific job and hopefully will match the needs of the designer, (2) the presentation of a compendium of solutions that can be matched or extrapolated to the designer's needs, as produced by Hrones and Nelson, (3) the use of a computer program as presented by Freudenstein to synthesize certain linkages to satisfy specific input data, and (4) the presentation of a system of design graphs such as Tesar's from which a linkage can be derived.

As we have seen, there have been theoretical methods presented to obtain a solution, however, there have been very few practical examples. Many designers cannot afford the time to go through the complete solution even if they have a computer and programming ability available. Even the writing or modification of a computer program from one computer language to another can be very inefficient. In fact a time consuming portion of the work represented by this thesis was spent in getting four programs to run repeatedly.

The goal of this thesis was then to provide the designer with at least a family of accomplished solutions in the form of several useful paths which can be mechanized.

The results could not be organized into any form of design graphs, however, a compendium of solutions is presented which gives the designer flexibility in assembling alternative linkages to follow the paths considered. The necessary computer programs, as converted into the common Fortran IV language, are also presented for use in further investigations or for other needed input requirements.

CHAPTER II

THE BURMESTER PROBLEM AND SYNTHESIS

Part I: The Classical Burmester Problem

The classical planar Burmester problem involves finding points on a body that lie on a circle which is fixed in a stationary reference frame as the body is moved through a series of specified positions relative to the stationary frame. A circle is used because it is the planar locus which is most easily mechanized, in so much as circular motion is generated by the simple pin or revolute joint. The number of positions to be specified (commonly referred to in the literature as accuracy points) determines the complexity of the problem and the method of solution. If only three successive positions are specified for the body, then every point in the body has its three positions on some unique circle since it takes only three points in the plane to specify a circle (see Fig. 1). This circle can be found very easily by graphical methods as shown in Fig. 2. If four positions are specified, then there is a line or locus on the body, every point of which is a solution. That is, every point lies on some unique circle for the four positions. This line and a similar line, on which are found the centers of the circles, may be found through graphical means.

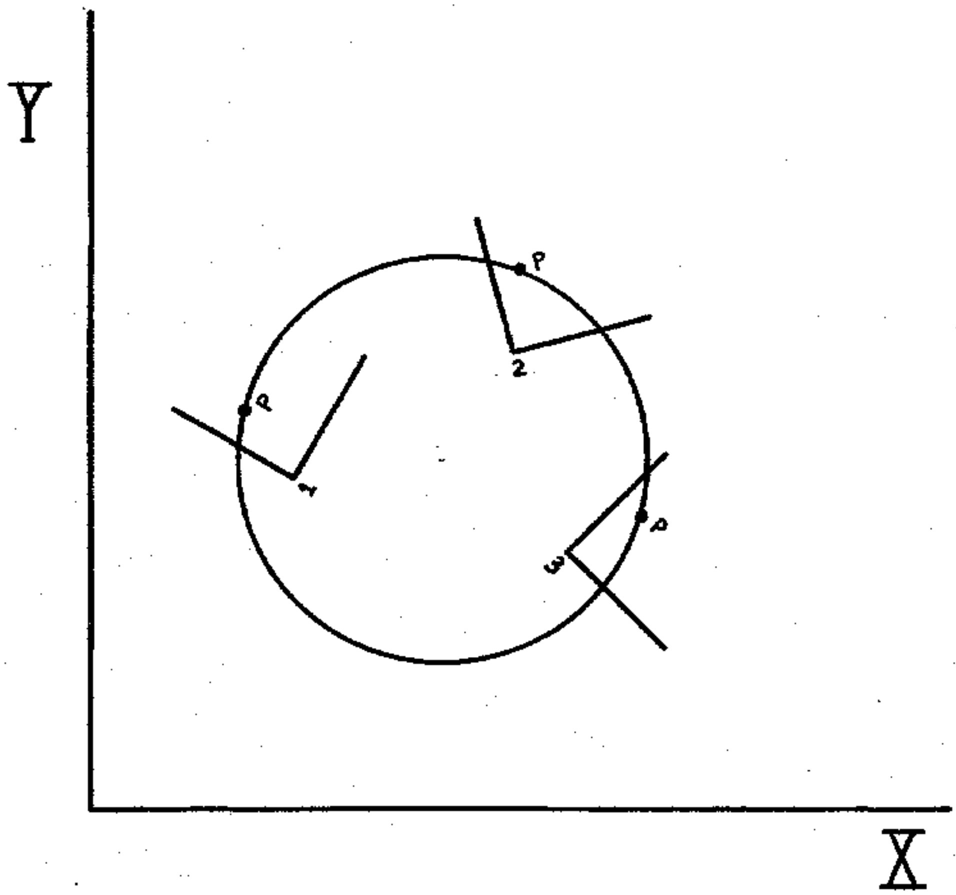

Figure 1. Circle Defined by Three Successive Planar Positions

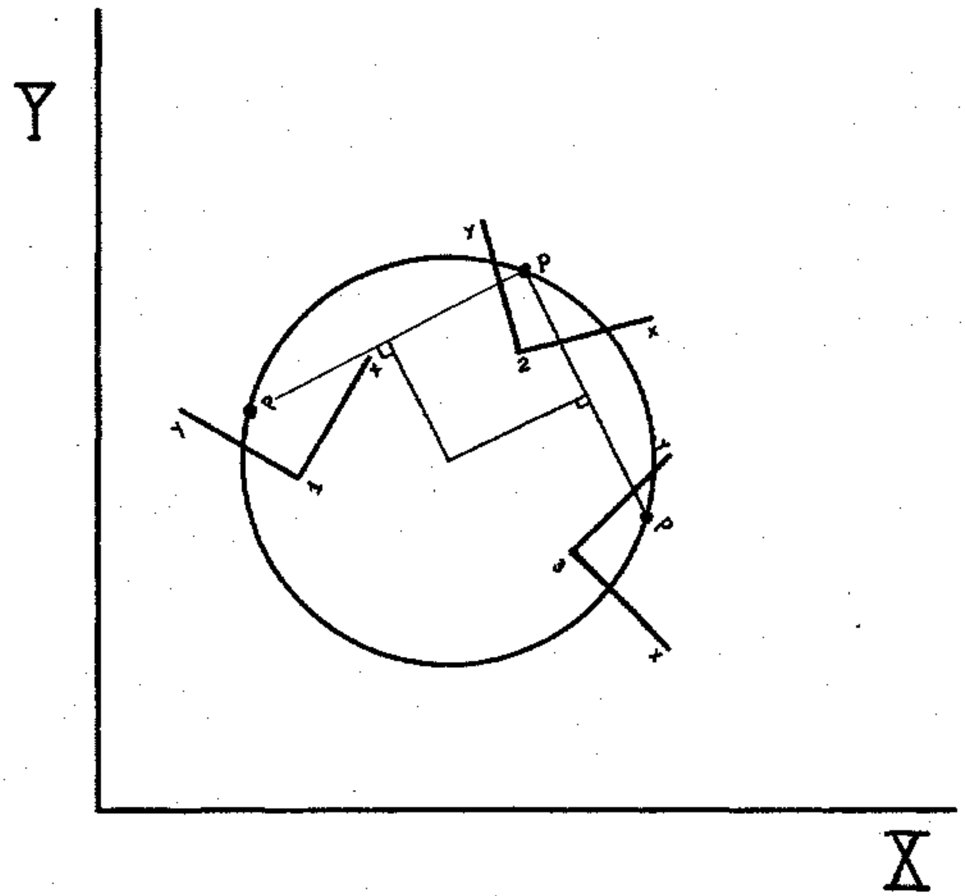

Figure 2. Construction to Define the Circle Associated with Three Planar Positions

Burmester found that certain isolated points could be found on a body such that they lie on a common circle for all of five specified positions. There are a maximum of four such points for each set of five positions.

Part II: The Three Dimensional Burmester Problem

The three dimensional Burmester problem involves finding points in a three dimensional body that lie on a spatial surface or locus--such as on a cylinder, on a sphere, or on some circle that is not necessarily parallel to one of the coordinate planes. The cylindrical surface is that surface which is generated by a cylindrical joint and the sphere is that surface which is generated by spherical joint (sometimes called a globular joint). Other surfaces have been investigated such as the line and plane [21,24,25] but this paper will concentrate on the cylinder, the sphere, and the circle because these are easily mechanized surfaces.

Part III: Mechanism Synthesis

The synthesis of a linkage using solutions of the Burmester problem can best be demonstrated by a planar example. The solution of the Burmester problem gives a maximum of four points and for each of these points, the location (xy) and radius (R) of a circle is established in the fixed reference plane. The center of the circle is called a center point while the point in the moving body is called a circle point. Figure 3 shows a notational scheme

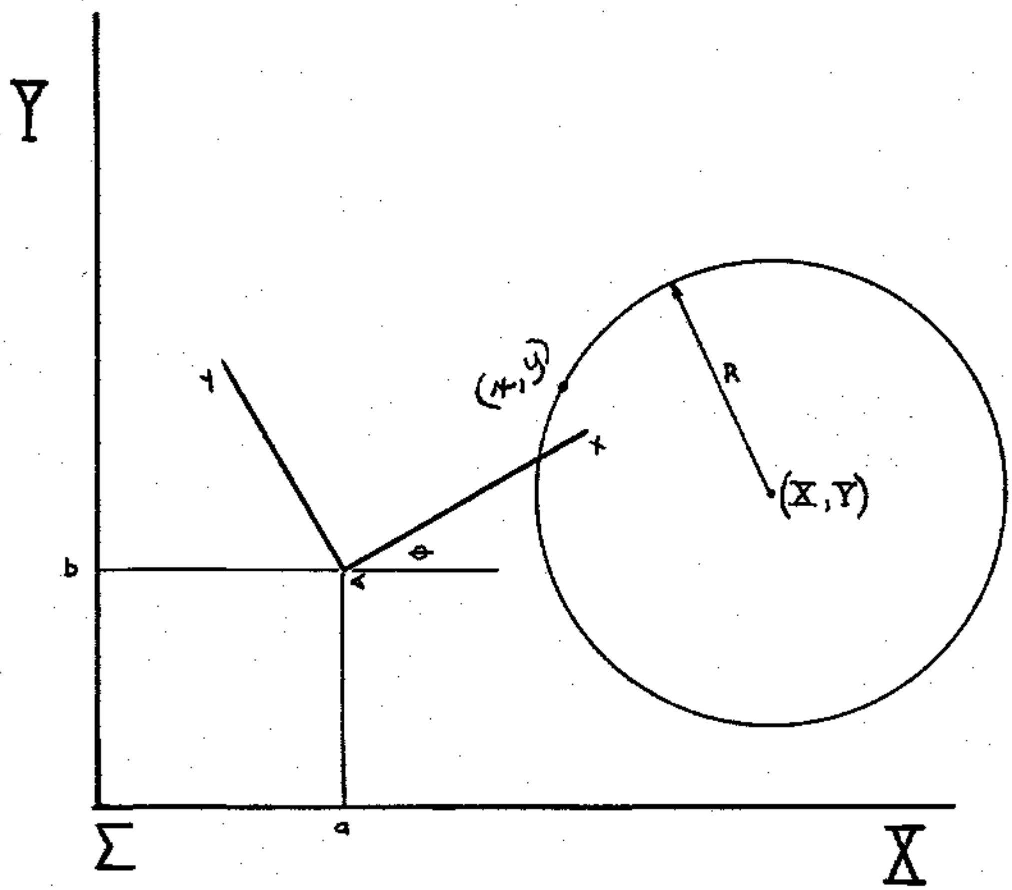

Figure 3. Notation for Burmester Points and Center Points

for the classical Burmester problem. The reference frame, σ, is the moving body, and five a's, b's, and θ's are given to specify the five positions of σ relative to the fixed reference frame, Σ. By making a circle point (x,y) and its corresponding center point (X,Y) the positions of revolute joints connected by a link of length equal to the radius of the circle, the moving body, σ, is connected to the fixed frame, Σ, by one link. By repeating the procedure with another circle and center point solution from the same input data, the moving body, σ, as defined by the two circle points, becomes the coupler link of a four-bar linkage. This four-bar linkage will carry the coupler through the five specified positions that were the input to the problem (see Figure 4).

The three dimensional problem uses the same approach except that the results from several different problems having the same input are combined to synthesize a linkage. For example, the linkage shown in Figure 5 was developed from one solution from the sphere problem and two solutions from the circle problem. This linkage will carry the coupler, σ, through the desired positions provided that the same input is given to the sphere and circle programs and that the linkage has suitable degrees of freedom determined by the well known Kutzbach-Greubler criterion [2]. The locations of the spherical (S) and revolute (R) joints are noted, and the spherical joints in the coupler may be observed not to restrict the relative motion of the coupler and the connecting

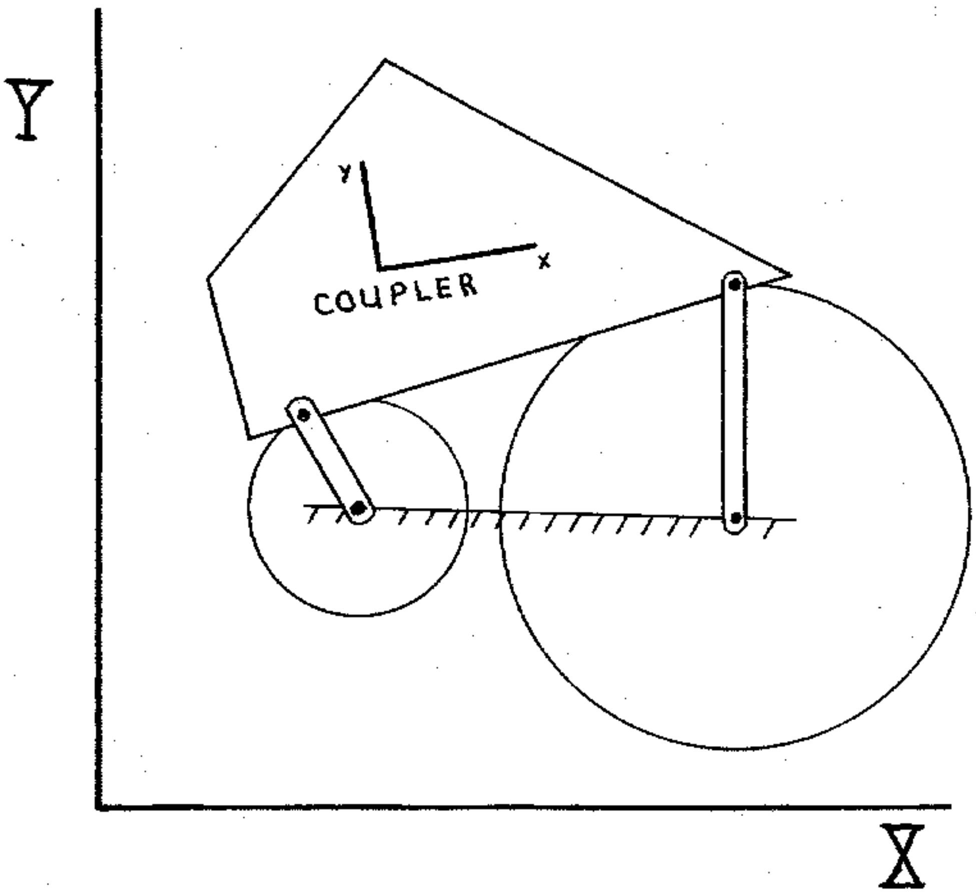

Figure 4. Four Bar Planar Linkage Based on Two Burmester Points

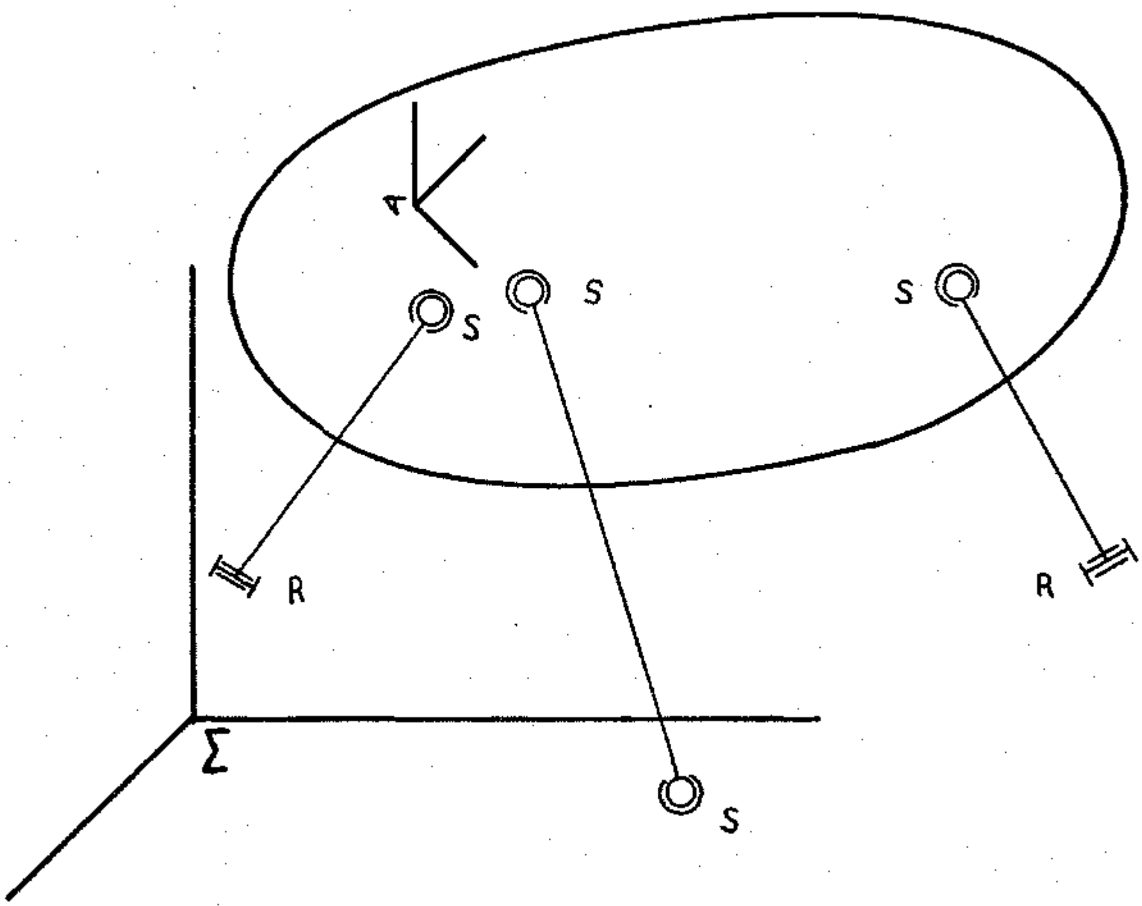

Figure 5. Five Link Spatial Mechanisms Based on Two Revolute Joints and One Spherical Joint

links in any way, except as to the point of attachment. One difficulty that exists when giving the same data to several problems is the fact that each problem is solved for a different number of specified positions. Williams [24] discusses this in his work and found the maximum number of arbitrary positions for the sphere problem as six, for the circle problem as four, and for the vertical cylinder problem as five. By specifying the initial position to coincide with the fixed frame, the equations are simplified. What happens to a linkage that has one part specified for four positions while another part is specified for those four positions plus two additional positions? While there is some tendency for the linkage to pass the coupler through the two extra positions, it cannot be assumed that it will. Since all the linkages contain revolute joints and the circle solution contains the minimum number of specified positions, four, all the linkages used in the study are four accuracy point mechanisms.

Mechanisms must also meet the mobility requirements as determined by the Kutzbach-Greubler criterion. The mechanism shown in Figure 5 has two degrees of freedom according to this criterion but one of these degrees of freedom is the unproductive axial rotation of the S-S link. The complete family of five link mechanisms which utilize the joints of interest and satisfy the Kutzbach-Greubler criterion will be introduced in the following chapter.

Part IV: Path and Position Synthesis

It has been shown that the input to the planar and the spatial Burmester problems are a set of positions of a body relative to some fixed reference. The output can be employed to synthesize a linkage that will carry the coupler through the input positions exactly. This approach can be used for the synthesis of path generating mechanisms by arranging the input positions so that they approximately specify a path. When path generation is desired, the relative angle between the coupler and the fixed frame loses importance and can be specified arbitrarily.

CHAPTER III

TYPES OF SPATIAL LINKAGES POSSIBLE

Solving the vertical cylinder, sphere, circle, and inverse circle problems for the same input data gives solutions that can be used to synthesize any one of six types of spatial linkages so as to satisfy the input requirements. These linkages are shown in Figure 6 and are listed below:

(a) A linkage with two R and one S joints on the fixed frame, and with three S joints on the moving frame;

(b) A similar linkage with two R and one C joints on the fixed frame;

(c) A linkage with two S and one R joints on the fixed frame, and two S and one R joints on the moving frame;

(d) A linkage with one R, one C, and one S joints on the fixed frame, and two S and one R joints on the moving frame;

(e) A linkage with three S joints on the fixed frame, and two R and one S joints on the moving frame;

(f) A linkage with two S and one C joints on the fixed frame, and two R and one S joints on the moving frame.

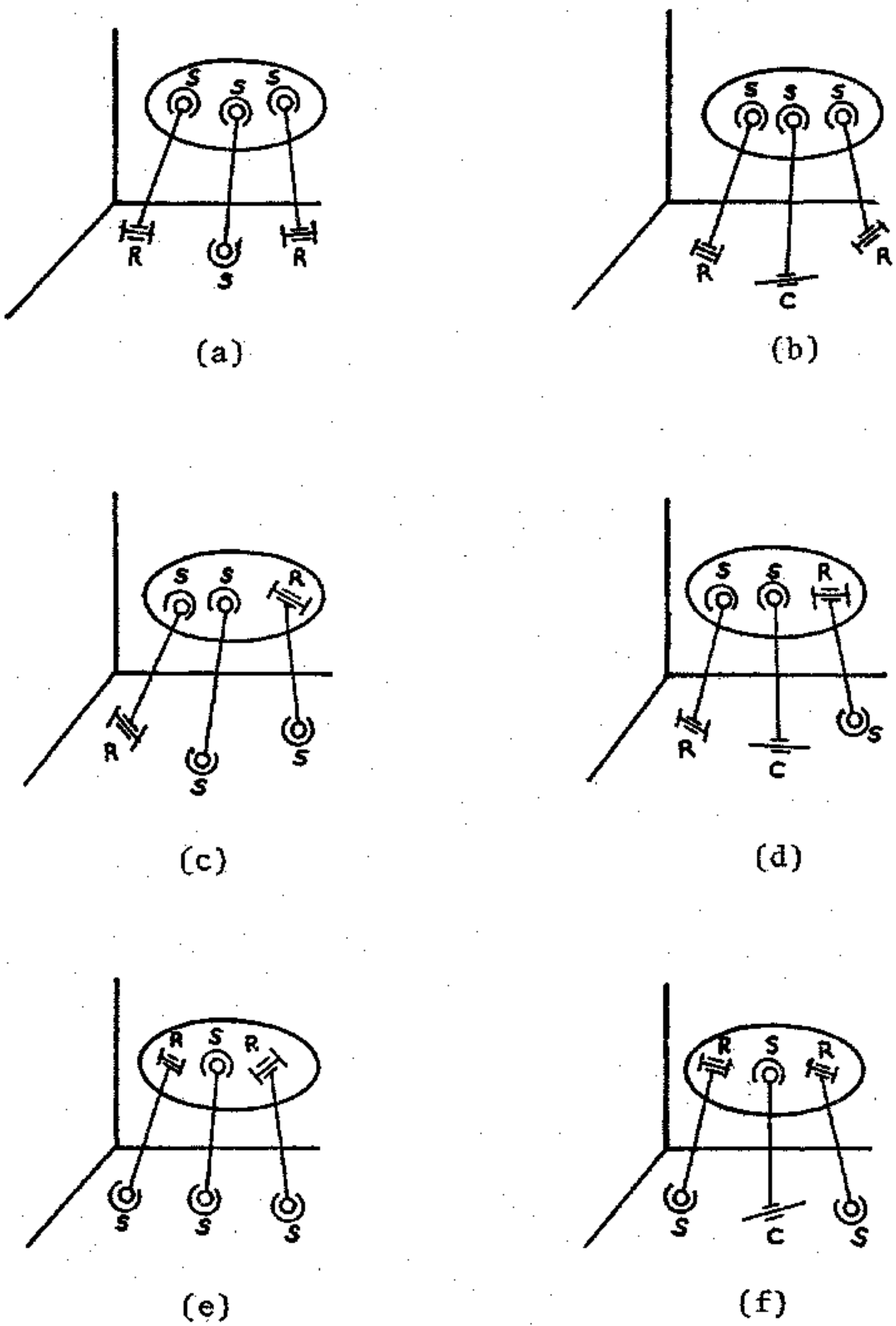

Figure 6. Types of Five Link Spatial Mechanisms Utilizing R, S, and C Joints Only

In the preceding list, R designates a revolute or pin joint, S designates a spherical or ball joint (which is also called a globular joint), and C designates a cylindrical joint. The two linkages (c) and (d) are similar to (a) and (b) in that they make use of the inverse circle problem to reverse the positions of the revolute joint from the fixed frame on (a) and (b) to the moving frame in (c) and (d). The last two linkages (e) and (f), are derived by using the inverse circle problem to transfer both R joints to the moving frame. Linkage (e) is not unique as it is the inverse of (a) and could be derived by reversing the fixed and moving frames of linkage (a).

All six of these linkage types have one effective degree of freedom. The three types with S-S links have two degrees of freedom with the extra motion provided in the axial spin of the S-S link, but that degree of freedom has no significance to the relative motion of the fixed and moving reference frames.

The vertical cylinder, sphere, circle, and inverse circle problems were solved for parameters describing a variety of input paths by a digital computer using the method of Williams [24]. In summary of the mathematics, this method takes the input data and writes the equation of the spatial surface (sphere, circle, and vertical cylinder) for each of the input positions. The initial position of the moving frame is specified to coincide with the fixed frame to simplify the equations. These equations are then solved simultaneously. This resulting data is used to solve a polynomial which had been derived from the compatibility relationships. Further, algebraic manipulation yields answers in a form that can be used for synthesis. Several things must be noted from this discussion of the mathematics. First, the initial position specified in the input must have zero values. Second, the uniqueness of the matrix must be preserved. This means that a parameter in the input (A, B, C, θ) cannot remain constant throughout the specified positions or, since the initial position always has a value of zero, the parameter cannot be zero for all of the input positions. This in effect rules out the use of this technique for a planar motion problem that utilized all of the input positions. However, the first value of an input parameter could be zero and the rest could be designated a constant value other than zero. This would yield a spatial mechanism whose coupler (moving frame) approximated planar motion for three accuracy positions. Since this could be solved with greater ease and with more accuracy points by planar methods, this limitation had minimal significance. The third point to be made regarding the mathematics is that when solving polynomial equations, imaginary roots may be encountered. These must be sorted out and are not presented.

CHAPTER IV

UTILIZING SYNTHESIS RESULTS IN MECHANISM DESIGN

There have been several limited attempts to provide design approaches to planar kinematic synthesis problems. Fruedenstein [3] provided a listing of four-bar mechanisms for function generation. These linkages have been optimized to have minimum error between the accuracy points. He provides a linkage for each of several common functions. Another approach was presented in the atlas of Hrones and Nelson of coupler point curves. This atlas provides the coupler curve of various points on the coupler of several four-bar linkages. The designer matches his required coupler curve as closely as possible to a curve in the atlas and obtains a linkage. This method's accuracy depends largely on eyesight and draftsmanship of the designer but is very quick and easy. Tesar has provided a series of design graphs for the specialized path generation of straight-line motion for a point on the coupler. One other method to aid the designer is exemplified by Freudenstein and Sandor [19]. This was the publication of a computer program which can save the designer a lot of time and is an invaluable aid for optimization. However, computer programs take a long time to write, to debug, and even to convert from one language to another and this may not be efficient or convenient.

The purpose of design aids are to reduce the time and effort spent in achieving a solution. A design graph is one type of design aid. The ideal design graph would be a graphical interpolator between the input data points for which exact solution were obtained. That is, the design graph maker would solve problems whose input varied by some increment and would present the data in a way so that the designer could devise an intermediate linkage whose input was between the solved data points. This ideal would allow the designer the greatest flexibility with the input data short of solving the problem exactly. Conversely, the presentation of a compendium of solutions places a great restriction on the designer as he must accept the input data that were chosen to be solved.

The first step in an effort to develop design graphs was the selection of the type of synthesis generator desired from the main types of path, position, or function generators. As was mentioned earlier, Burmester theory does not lend itself very well to the design of function generators. The second step was an attempt to classify path and position generators by the type of input motion. Since a group of positions by their definitions are arbitrary and do not have a functional relationship, position synthesizers were not studied for possible reduction to design graphs. The input motions for path generators was divided into two categories. The first category consisted of motion that performed some task such as the displacement of the coupler around some object. The second category consisted of motion along a path described by some mathematical function. A special case of the second group was the type of linkage where the y coordinate was some function of x and the z coordinate was some other function of x. This gave a linkage whose projection of the coupler's origin on the x-y plane generated a functional path such as $y = x^2$ while the projection on x-z plane generated a path such as $z = x^3$.

The third step in the attempt to develop a system of design graphs was thesolving of a significant collection of problems with the aid of a computer. The input data was varied in an orderly manner to search for some relationship between the input parameter and output parameters. Figure 7 shows the notation used to define input parameters and Figures 8, 9, 10, and 11 show the notation for the output from the vertical cylinder, sphere, circle, and inverse circle programs, respectively.

A set of input data was prepared where the x-coordinate varied by a small increment. It was found that the values of D in the output of the vertical cylinder program varied in an approximately linear relationship with the x of the input. However, the E, R, x and y values of the output varied in an unpredictable manner. This was to be expected. The D output

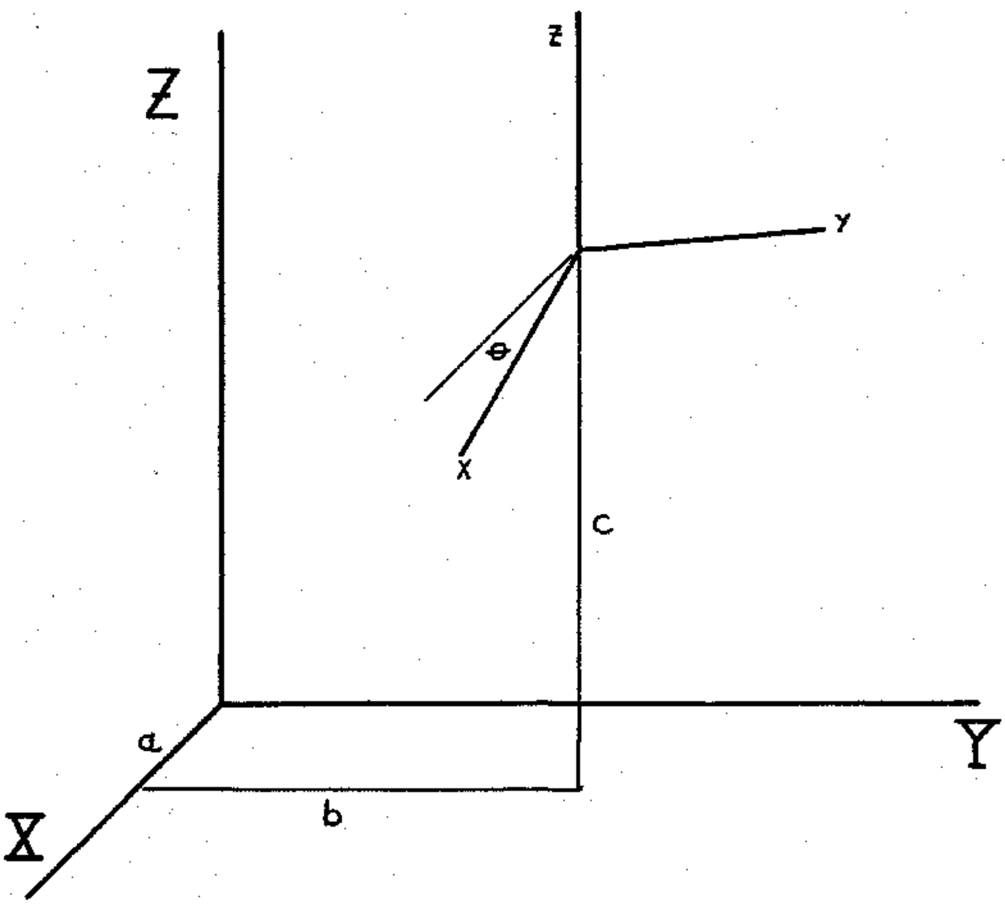

Figure 7. Input Parameters to Define a Position of the Moving Frame

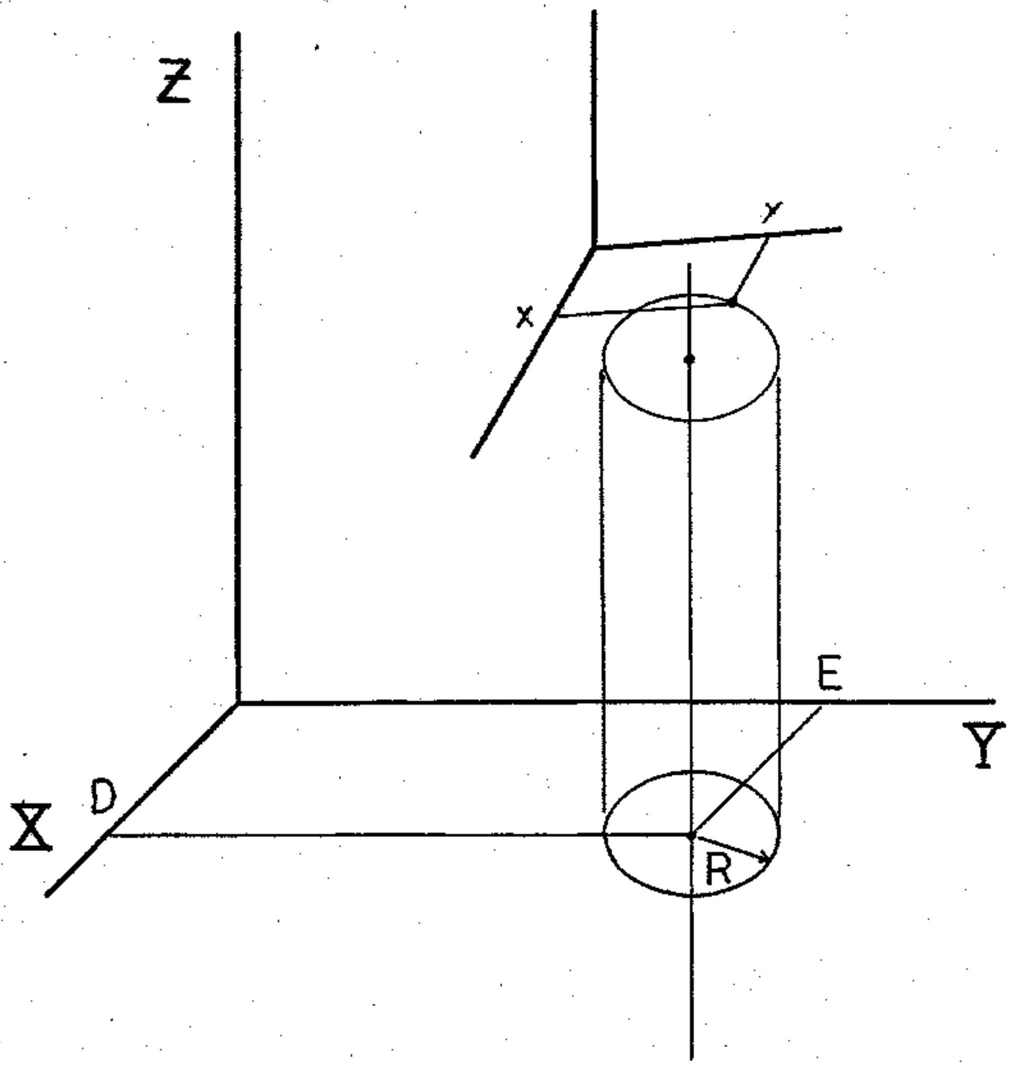

Figure 8. Output Parameters Defining a Cylindrical Joint

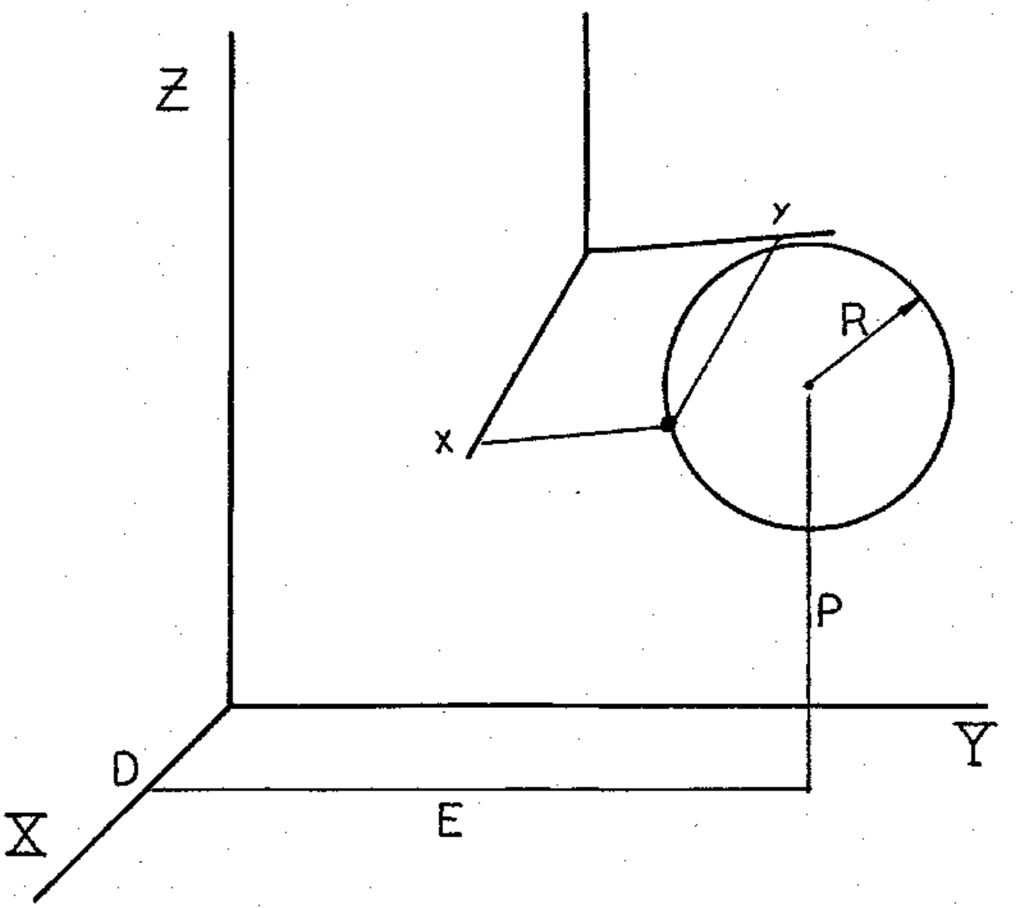

Figure 9. Output Parameters Defining a Spherical Joint

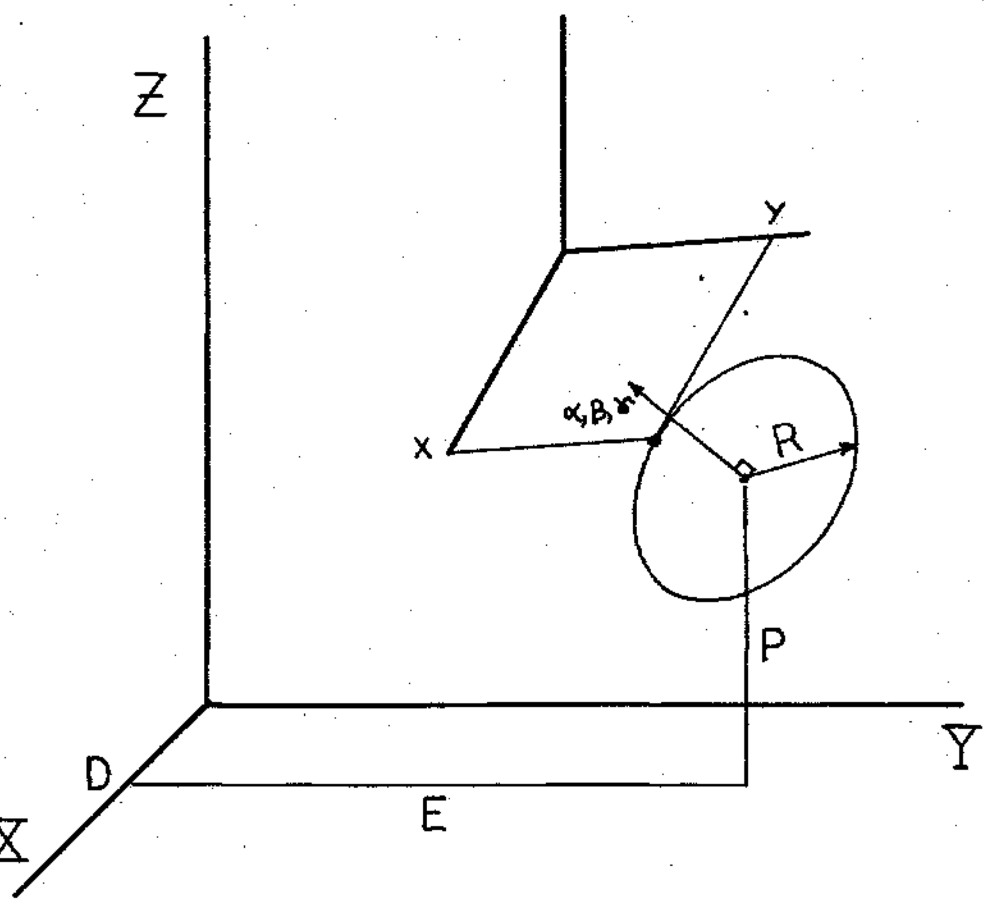

Figure 10. Output Parameters Defining the Circle of a Revolute Joint

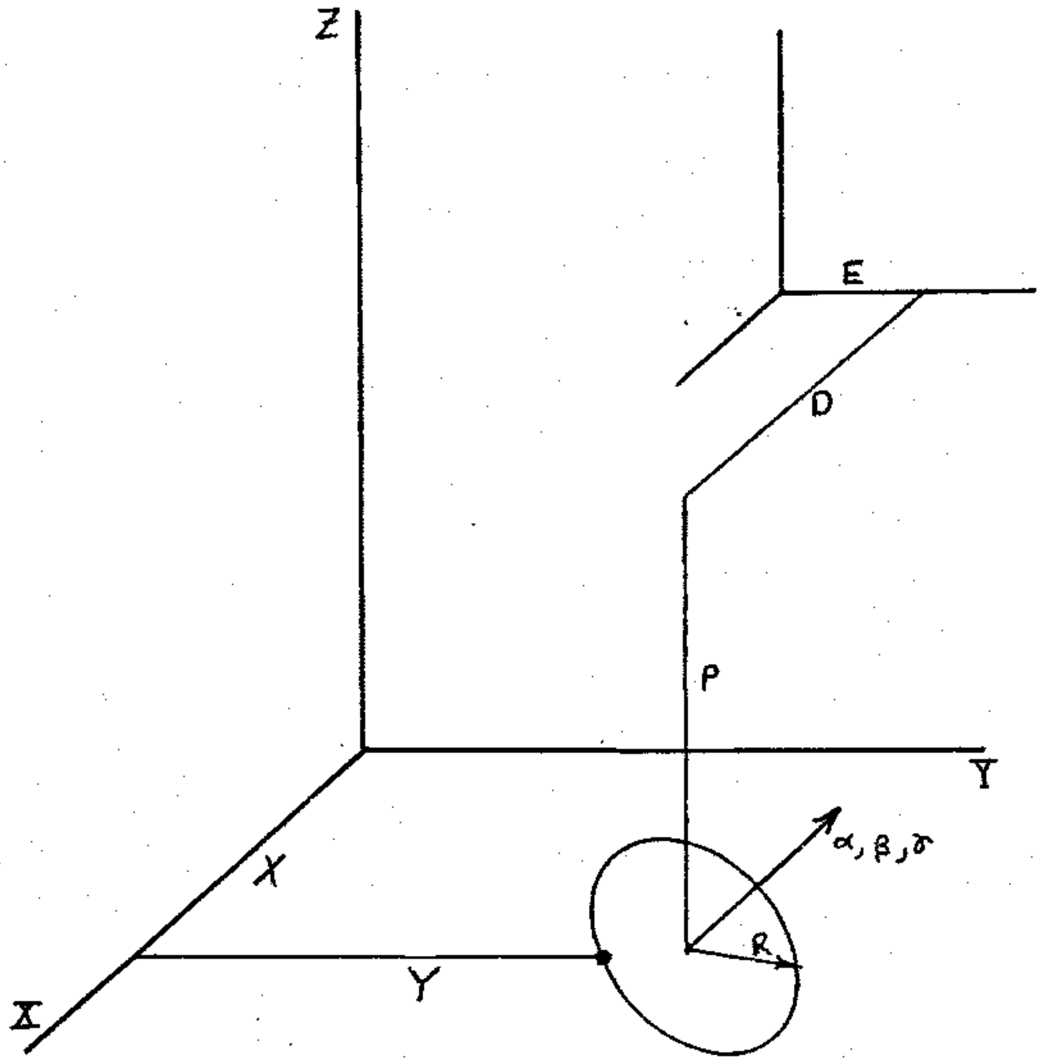

Figure 11. Output Parameters Defining the Inverse Circle for a Revolute Joint

variable is the x-coordinate of the center of the vertical cylinder (the axis of the cylindrical joint) in the fixed frame. Physically, the x-coordinate of the input positions must have some bearing on the general position of the vertical cylinder in the x direction. The other output parameters do not have such a strong physical relationship and they do not have a linear mathematical relationship as the solution of the vertical cylinder problem involves the solving of a 4 x 5 matrix and a fourth-order algebraic equation.

The major problem was the complexity of the three dimensional linkage in terms of the total number of parameters involved. A linkage such as that shown in Figure 12 requires twenty-two parameters for definition whereas a planar four bar linkage with a coupler point requires just eleven parameters.

The spatial solutions generated in this work and their linkage parameters thus do not lend themselves to graphical presentation. Nonetheless, families of several types of specific coupler paths of potential usefulness to the designer will be presented. The Burmester solutions (and thus the associated possible linkage parameters) for each path will then be presented for use of the designer wanting to synthesize a spatial mechanism.

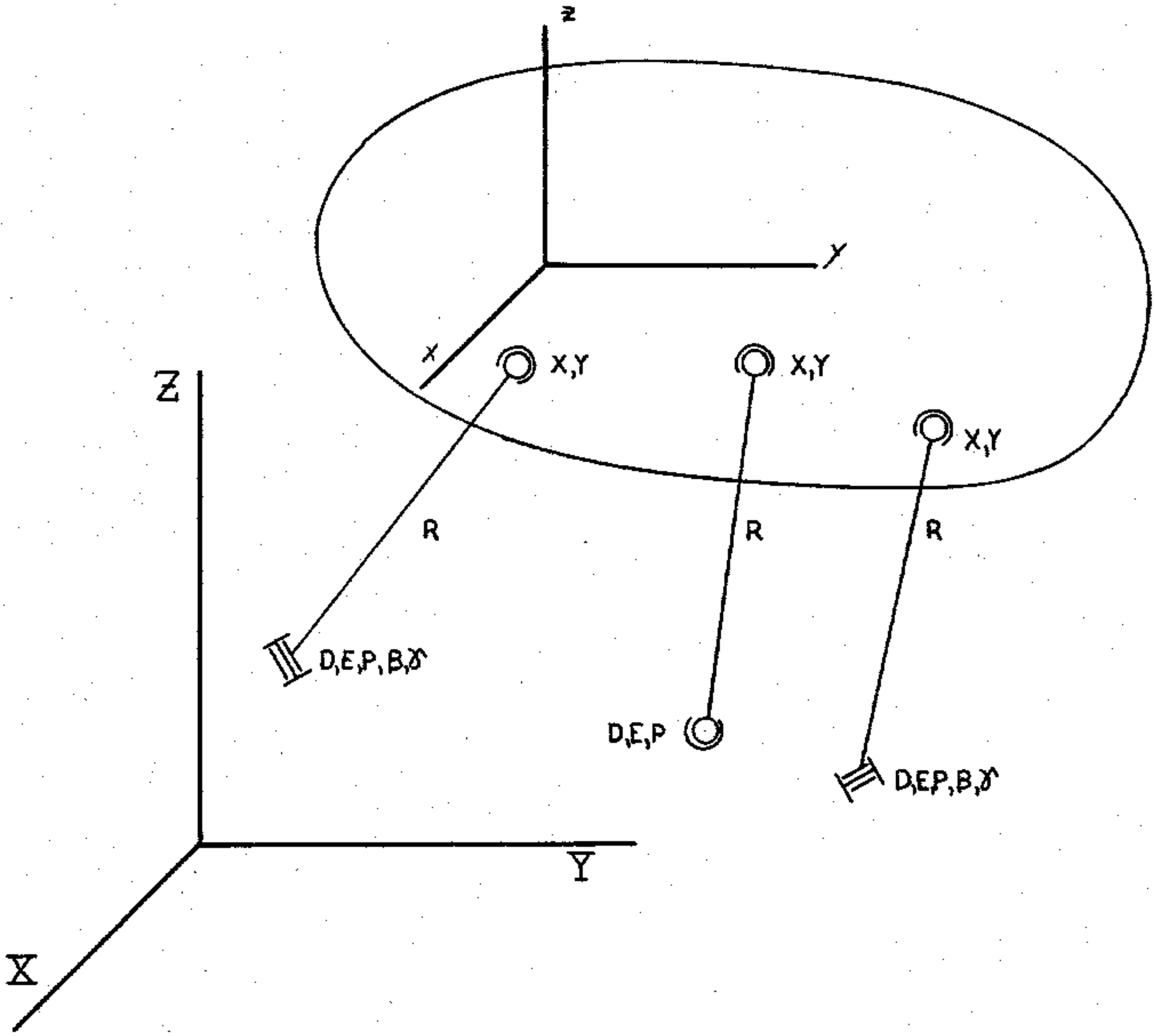

Figure 12. Parameter Used in Defining the Joints of a Five Link Mechanism

CHAPTER V

PRESENTATION AND USE OF THE SOLUTIONS

This chapter explains the organization scheme of the presentation of the solutions in the appendix, a description of the purpose of each of the paths or tasks chosen to be solved, and an example of how a solution is used to design a linkage.

The solutions are organized by a number-letter-number system. The first number refers to a unique task or set of input data used in each of the four computer programs to develop a family of solutions. The letter refers to the type of solution--whether the solution resulted from the use of the vertical cylinder, sphere, circle, or inverse circle computer programs. The second number denotes each of the solutions generated by a computer program. For example, if the data for the first task develops two sphere solutions and three vertical cylinder solutions, they would be designated as follows: 1-S-1 and 1-S-2, and 1-VC-1, 1-VC-2, and 1-VC-3. Each of the sphere solutions could be interchangeable with another of the sphere solutions, etc. All of the solutions for a given task are then assembled on a single chart in Appendix A.

Tasks for Which Mechanisms can be Synthesized

Each task represents an appropriate and useful set of precision points defining a coupler motion. Some tasks define an arbitrary coupler path and others define a functional relationship between the coordinates of the coupler link. The description of each of the tasks selected is as follows:

TASK NO. 1: This motion maintains a functional relationship between the coordinates of the origin of the coupler in both the x-z plane and the x-y plane. The function in the x-z plane is that the z coordinate (C) is the square of the x coordinate (A). The function in the x-y plane is that the y coordinate (B) is the logarithm of the x coordinate (A).

TASK NO. 2: This motion is designed to carry the coupler around some object in the x-y plane with a height measured in the z direction. This object would be centered at approximately x = 3 and y = 5 in the x-y plane.

TASK NO. 3: This motion is similar to Task No. 2 except that the displacements in the z direction (C) are different and the angular displacements (θ) are different.

TASK NO. 4: This motion gives a functional relationship where the y coordinate of the origin of the coupler (B) is the logarithm of the x coordinate (A).

TASK NO. 5: This motion gives the same functional relationship as Task No. 1. The θ values of the input have been varied to give alternate mechanisms.

TASK NO. 6: This motion gives the same functional relationship as Task No. 1 except the order of the input positions has been changed. The effect of this is that the "extra" positions needed to solve the circle and vertical cylinder problems are at the end of the coupler's path. This should make the mechanism more accurate as a function generator but over a shorter interval.

TASK NO. 7: This motion would carry the coupler in a short approximately spiral path from the origin.

TASK NO. 8: This motion gives the same type of path as Task No. 7 except that the displacements in the y direction have been magnified.

TASK NO. 9: This motion gives the same type of path as Tasks No. 7 and 8 except that the displacements in the y direction have been additionally magnified.

TASK NO. 10: This motion gives the same type of path as Tasks No. 7, 8, and 9 except that the displacements in the y direction have been additionally magnified.

In general, the families of Burmester solutions developed for each task can be used to develop one or more mechanisms to carry out the coupler displacements defined by the tasks.

<u>Example</u>

Using Task No. 3, a linkage can be derived that will carry an object around some obstacle. Choosing a linkage of the type shown in Figure 6c, a solution for the sphere problem, a solution from the circle problem, and a solution from the inverse circle problem are required. Figure 13 shows all six positions designated. Positions 0, 1, 2, and 3 are the only positions that the coupler will be sure to pass through. Positions 4 and 5 are chosen to help guide the linkage on the path between the accuracy points. Figure 14 shows the derived linkage in position 2.

The direction numbers (α,β,γ) are a set of numbers associated with the normal to the plane of a circle, such that a displacement along the axis of a revolute (or pin) joint has its x component proportional to α, its y component proportional to β, and its z component proportional to γ.

Further note that for this four-degree-of-freedom coupler motion, that any connecting link can be shifted vertically for convenience. Thus in Figure 14 it is seen for the link having a pin joint in the moving frame, that the ball joint in the fixed frame has been placed 1.33 units above the XY plane rather than to have placed the pin joint 1.33 units below the xy plane of the moving frame.

Conclusions

Four computer programs, which solve the sphere, vertical cylinder, circle, and inverse circle problems in three dimensions [24], were converted to the FORTRAN IV computer language and are listed in Appendix B. These

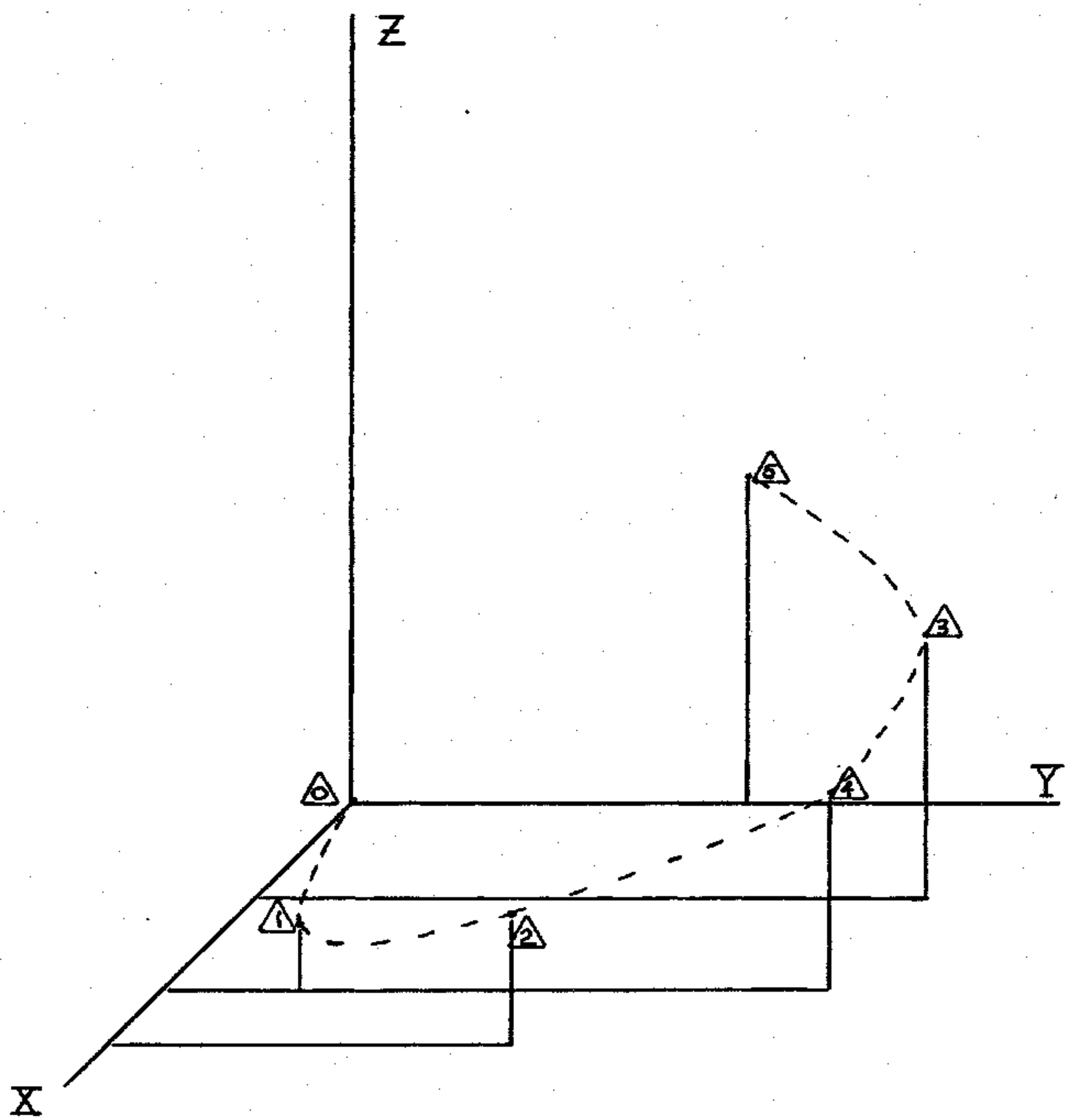

Figure 13. Successive Locations of the Origin of the Moving Frame in Performing Task No. 3

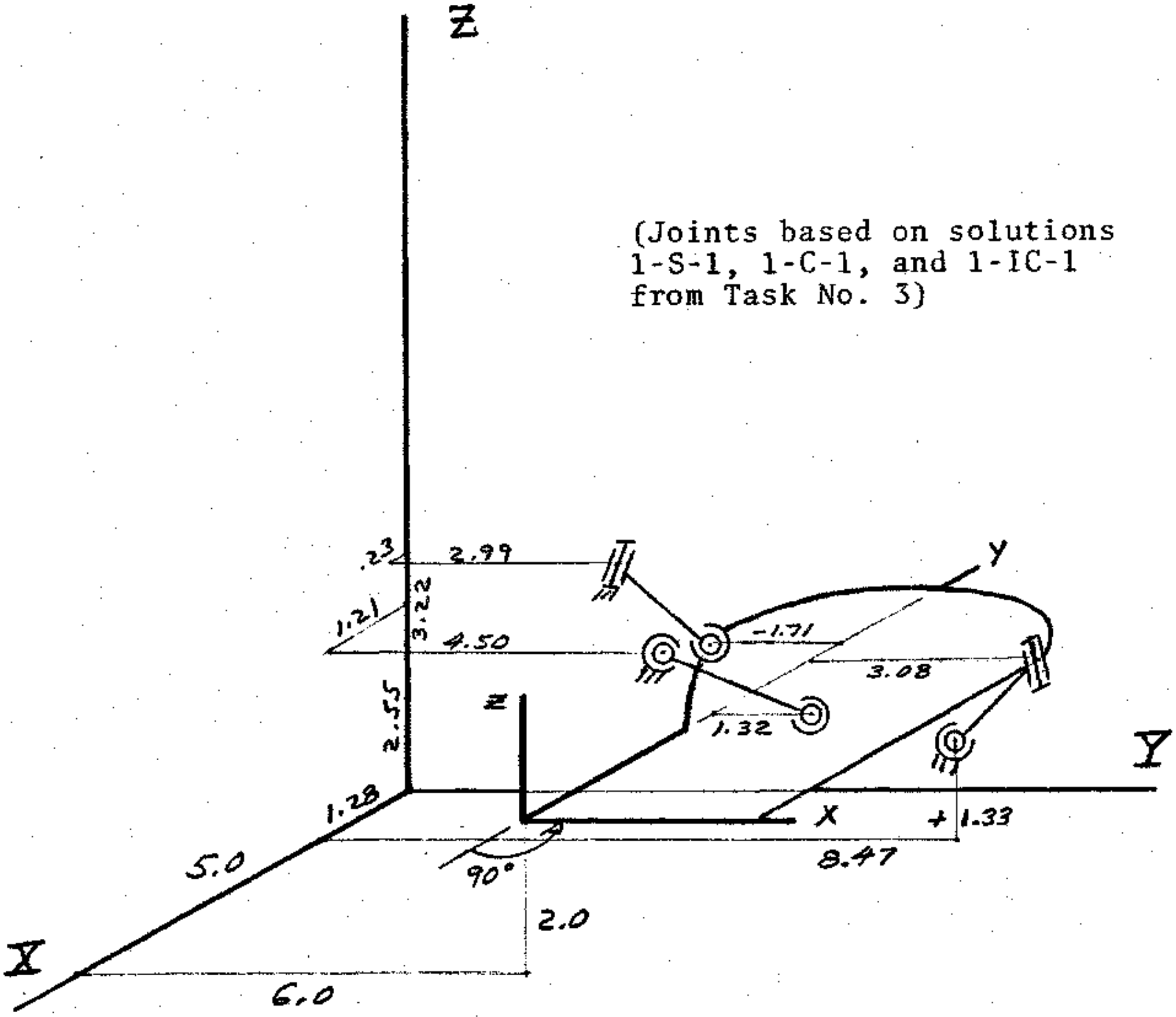

Figure 14. Example Linkage Performing Task No. 3 and Shown in Position No. 2

programs were used to develop the body of results in Appendix A which can be used to develop numerous spatial linkages which perform some task. The ten tasks (or three-dimensional motions) which can be performed from the results tabulated in this study were carefully selected for possible usefulness.

The net accomplishment is hopefully that spatial mechanisms following certain useful paths can now be synthesized by designers not otherwise able to justify the computational effort of synthesizing directly from input data.

A further program can be recommended of identifying additional useful motions and generating the subsequent solutions which permit their mechanization.

APPENDICES

APPENDIX A

INPUT PARAMETERS AND SOLUTIONS FOR EACH OF THE TEN DISPLACEMENT TASKS DEFINED IN CHAPTER V

Task Number 1

INPUT

Position No.	A	B	C	θ
0	0.0	0.0	0.0	0.0
1	1.1	0.041	1.21	10.0
2	1.3	0.114	1.69	20.0
3	1.5	0.176	2.25	30.0
4	1.2	0.079	1.44	15.0
5	1.4	0.146	1.96	25.0

Sphere problem uses positions 0 through 5. Vertical cylinder problem uses positions 0 through 4. Circle problems use positions 0 through 3.

OUTPUT

Type Joint	Solution No.	Radius R	Location of Joint on Fixed Frame						Location on Moving Frame	
			D	E	P	α	β	γ	x	y
Sphere	1	7.5248	-0.7111	-0.2383	5.6144				3.3874	-3.1196
	2	27.1518	-19.6839	-3.3497	20.4725				-1.9781	-1.2046
Vertical Cylinder	1	0.4993	0.5619	1.1721					0.1141	1.1792
	2	3.4679	0.2545	2.0441					-0.32081	-0.4474
	3	4.8683	0.8440	3.6124					0.5431	6.3846
Circle	1	8.8442	-5.3054	2.7537	6.4535	1.000	2.8793	-0.9564	-0.5867	-1.0287
	2	33.644	-1.9700	32.5046	1.1956	1.000	-0.0051	-1.0991	-3.4551	-1.0056

Type Joint	Solution No.	Radius R	Location on Fixed Frame		Location of Joint on Moving Frame					
			X	Y	D	E	P	α	β	γ
Inverse Circle	1	3.7702	0.2960	-0.7643	1.5371	2.0039	-2.3384	1.000	-1.3702	-1.1401
	2	9.6038	4.7957	- .7583	4.5911	8.8411	- .2060	1.000	- .0016	-1.0693

Task Number 2

INPUT

Position No.	A	B	C	θ
0	0.0	0.0	0.0	0.0
1	4.0	2.0	1.0	10.0
2	5.0	6.0	2.0	20.0
3	2.0	10.0	2.0	30.0
4	4.0	10.0	3.0	40.0
5	0.0	6.0	0.0	50.0

Sphere problem uses positions 0 through 5. Vertical cylinder problem uses positions 0 through 4. Circle problems use positions 0 through 3.

OUTPUT

Type Joint	Solution No.	Radius R	Location of Joint on Fixed Frame						Location on Moving Frame	
			D	E	P	α	β	γ	x	y
Sphere	1	2.2389	-5.5483	2.3949	9.7481				-4.8204	0.2774
Vertical Cylinder	1	3.4907	-5.7241	5.8124					-7.0520	2.8332
Circle										

Type Joint	Solution No.	Radius R	Location on Fixed Frame		Location of Joint on Moving Frame					
			X	Y	D	E	P	α	β	γ
Inverse Circle	1	1.7984	-15.1922	9.1366	-16.3897	9.1359	-1.3417	1.000	1.2872	-0.8933

Task Number 3

INPUT

Position No.	A	B	C	θ
0	0.0	0.0	0.0	0.0
1	4.0	2.0	1.0	30.0
2	5.0	6.0	2.0	90.0
3	2.0	10.0	4.0	180.0
4	4.0	10.0	3.0	180.0
5	0.0	6.0	5.0	270.0

Sphere problem uses positions 0 through 5. Vertical cylinder problem uses positions 0 through 4. Circle problems use positions 0 through 3.

OUTPUT

Type Joint	Solution No.	Radius R	Location of Joint on Fixed Frame D	E	P	α	β	γ	Location on Moving Frame x	y
Sphere	1	1.5774	1.2067	4.5007	2.5504				1.3185	2.9272
	2	10.4867	.0389	9.3359	-8.4316				-3.0047	- .6993
Vertical Cylinder	1	8.4331	3.8905	-2.6676					-0.8905	4.3706
Circle	1	4.1790	0.2347	2.9998	3.2225	1.0000	-1.1462	-1.2493	-1.7109	4.8148
	2	12.9930	3.0604	-6.5227	2.7406	1.0000	0.0139	-2.5648	-4.1151	3.9568

Type Joint	Solution No.	Radius R	Location on Fixed Frame X	Y	Location of Joint on Moving Frame D	E	P	α	β	γ
Inverse Circle	1	4.6032	1.2820	8.4694	3.0754	4.4424	-1.3254	1.0000	1.1462	-2.1293
	2	21.8363	3.9923	7.0330	9.8304	-13.6688	3.7654	1.0000	0.0083	-1.5045

Task Number 4

INPUT

Position No.	A	B	C	θ
0	0.0	0.0	0.0	0.0
1	1.1	0.041	1.0	10.0
2	1.3	0.114	1.0	20.0
3	1.5	0.176	1.0	30.0
4	1.2	0.079	1.0	15.0
5	1.4	0.146	1.0	25.0

Sphere problem uses positions 0 through 5. Vertical cylinder problem uses positions 0 through 4. Circle problems use positions 0 through 3.

OUTPUT

Type Joint	Solution No.	Radius R	Location of Joint on Fixed Frame D	E	P	α	β	γ	Location on Moving Frame x	y
Sphere	1	1.9645	1.5216	1.4985	-0.7096				-0.2855	1.1992
	2	4.0558	-1.7239	1.6448	3.1369				0.7175	0.8388
Vertical Cylinder	1	0.4993	0.5619	1.1721					0.1142	1.1792
	2	3.4679	0.2544	2.0441					-0.3208	-0.4474
	3	4.8683	0.8440	3.6124					0.5430	6.3846
Circle	1	0.6714	.8446	1.1449	0.4952	1.0000	-0.0014	-0.9061	.3957	1.0811
	2	0.6705	0.6107	1.1920	0.4987	1.0000	0.1656	-0.8993	0.1625	1.1902
	3	0.6917	0.3709	1.0504	0.4709	1.0000	0.3522	-0.8917	-0.1076	1.2167
	4	0.6745	0.4862	1.1444	0.4905	1.0000	0.2568	-0.8956	0.0287	1.2152

Type Joint	Solution No.	Radius R	Location on Fixed Frame X	Y	Location of Joint on Moving Frame D	E	P	α	β	γ
Inverse Circle	1	0.6725	0.9166	1.2426	0.4695	1.1599	-0.4956	1.0000	- .0984	- .8858
	2	0.6722	0.7827	1.2205	0.3314	1.1586	-0.4943	1.0000	-0.1872	-0.8896

Task Number 5

INPUT

Position No.	A	B	C	θ
0	0.0	0.0	0.0	0.0
1	1.1	0.041	1.21	5.0
2	1.3	0.114	1.69	7.0
3	1.5	0.176	2.25	9.0
4	1.2	0.079	1.44	6.0
5	1.4	0.146	1.96	8.0

Sphere problem uses positions 0 through 5. Vertical cylinder problem uses positions 0 through 4. Circle problems use positions 0 through 3.

OUTPUT

Type Joint	Solution No.	Radius R	Location of Joint on Fixed Frame D	E	P	α	β	γ	Location on Moving Frame x	y
Sphere	1	47.9844	28.0540	-20.0053	41.7729				49.7920	-29.2258
Vertical Cylinder	1	0.9480	- 0.3409	9.0278					- 0.3535	9.9345
Circle	1	11.827	-12.5714	- 8.6245	10.8615	1.0	2.7376	-0.2314	- 8.5468	-11.0131

Type Joint	Solution No.	Radius R	Location on Fixed Frame X	Y	Location of Joint on Moving Frame D	E	P	α	β	γ
Inverse Circle	1	9.7311	- 6.1556	-11.3143	-1.8604	-7.6379	-7.9202	1.0000	37.2709	17.8423

Task Number 6

INPUT

Position No.	A	B	C	θ
0	0.0	0.0	0.0	0.0
1	1.1	0.041	1.21	10.0
2	1.2	0.079	1.44	15.0
3	1.3	0.114	1.69	20.0
4	1.4	0.146	1.96	25.0
5	1.5	0.176	2.25	30.0

Sphere problem uses positions 0 through 5. Vertical cylinder problem uses positions 0 through 4. Circle problems use positions 0 through 3.

OUTPUT

Type Joint	Solution No.	Radius R	Location of Joint on Fixed Frame D	E	P	α	β	γ	Location on Moving Frame x	y
Sphere	1	7.5248	-0.7111	-0.2383	5.6144				3.3874	-3.1196
	2	27.1518	19.6839	-3.3497	20.4725				-1.9781	-1.2046
Vertical Cylinder	1	0.4735	0.5216	1.1681					0.0743	1.1945
	2	7.0919	0.4265	0.6103					-0.2817	-0.4533
	3	6.8856	0.5928	3.1771					0.4451	9.1017
Circle	1	8.1447	-5.0095	2.1938	6.0617	1.0000	3.2193	-0.9368	-0.5696	-0.9492
	2	30.143	-1.8755	29.038	1.0260	1.0000	- .0099	-1.1061	-3.3066	-1.0538

Type Joint	Solution No.	Radius R	Location on Fixed Frame X	Y	Location of Joint on Moving Frame D	E	P	α	β	γ
Inverse Circle	1	3.3921	0.1017	-0.7363	1.1216	1.8459	-1.9489	1.0000	-1.2701	-1.1594
	2	7.9520	4.3166	-0.9452	4.0825	7.0001	-0.2255	1.0000	- .0048	-1.0903

Task Number 7

INPUT

Position No.	A	B	C	θ
0	0.0	0.0	0.0	0.0
1	1.5	0.5	1.0	10.0
2	2.0	1.0	1.4	90.0
3	0.0	2.0	1.8	180.0
4	2.0	0.75	1.2	45.0
5	1.0	1.5	1.6	135.0

Sphere problem uses positions 0 through 5. Vertical cylinder problem uses positions 0 through 4. Circle problems use positions 0 through 3.

OUTPUT

Type Joint	Solution No.	Radius R	Location of Joint on Fixed Frame						Location on Moving Frame	
			D	E	P	α	β	γ	x	y
Sphere	1	14.4665	8.9516	10.7869	-16.8284				-0.7521	0.0576
	2	4.7616	3.0787	0.8205	- 4.8482				-1.6724	1.1370
Vertical Cylinder	1	1.5961	1.1484	1.3119					0.0852	2.0604
Circle	1	1.1802	0.4634	1.0332	0.9942	1.0	19.5976	11.8440	0.2282	0.4447

Type Joint	Solution No.	Radius R	Location on Fixed Frame		Location of Joint on Moving Frame					
			X	Y	D	E	P	α	β	γ
Inverse Circle	1	2.6573	1.8324	0.5204	0.1523	2.5422	-0.3883	1.0	0.4900	-1.7749

Task Number 8

INPUT

Position No.	A	B	C	θ
0	0.0	0.0	0.0	0.0
1	1.5	0.6	1.0	10.0
2	2.0	1.2	1.4	90.0
3	0.0	2.4	1.8	180.0
4	2.0	0.9	1.2	45.0
5	1.0	1.8	1.6	135.0

Sphere problem uses positions 0 through 5. Vertical cylinder problem uses positions 0 through 4. Circle problems use positions 0 through 3.

OUTPUT

Type Joint	Solution No.	Radius R	Location of Joint on Fixed Frame D	E	P	α	β	γ	Location on Moving Frame x	y
Sphere	1	4.3312	2.6678	0.8013	-3.9999				-1.6295	1.3573
	2	0.4333	0.2026	1.5866	0.9080				-0.8303	0.1426
Vertical Cylinder	1	1.9299	1.4403	1.1550					0.0442	2.4830
Circle	1		0.3852	1.2156	0.9780	1.0	8.1985	-6.5840	0.2126	0.4513

Type Joint	Solution No.	Radius R	Location on Fixed Frame X	Y	Location of Joint on Moving Frame D	E	P	α	β	γ
Inverse Circle	1	2.3676	1.7405	0.6805	0.1691	2.3618	-0.5564	1.0	0.3641	-1.7238

Task Number 9

INPUT

Position No.	A	B	C	θ
0	0.0	0.0	0.0	0.0
1	1.5	0.7	1.0	10.0
2	2.0	1.4	1.4	90.0
3	0.0	2.8	1.8	180.0
4	2.0	1.05	1.2	45.0
5	1.0	2.1	1.6	135.0

Sphere problem uses positions 0 through 5. Vertical cylinder problem uses positions 0 through 4. Circle problems use positions 0 through 3.

OUTPUT

Type Joint	Solution No.	Radius R	Location of Joint on Fixed Frame D	E	P	α	β	γ	Location on Moving Frame x	y
Sphere	1	14.0354	9.8471	9.2771	-19.0028				-0.8789	0.2248
	2	3.8949	2.2633	0.8608	- 3.1412				-1.5667	1.5695
Vertical Cylinder	1	1.5610	1.3062	1.3724					1.8050	2.2541
Circle	1	1.3522	0.2831	1.4033	0.9588	1.0	5.0550	-5.0959	0.1981	0.4535

Type Joint	Solution No.	Radius R	Location on Fixed Frame X	Y	Location of Joint on Moving Frame D	E	P	α	β	γ
Inverse Circle	1	2.1537	1.6521	0.8446	0.1588	2.2444	-0.6704	1.000	0.2664	-1.6713

Task Number 10

INPUT

Position No.	A	B	C	θ
0	0.0	0.0	0.0	0.0
1	1.5	0.8	1.0	10.0
2	2.0	1.6	1.4	90.0
3	0.0	3.2	1.8	180.0
4	2.0	1.2	1.2	45.0
5	1.0	2.4	1.6	135.0

Sphere problem uses positions 0 through 5. Vertical cylinder problem uses positions 0 through 4. Circle problems use positions 0 through 3.

OUTPUT

Type Joint	Solution No.	Radius R	Location of Joint on Fixed Frame D	E	P	α	β	γ	Location on Moving Frame x	y
Sphere	1	14.3262	10.4293	9.0594	-20.3864				-0.9071	0.3000
	2	3.4632	1.8903	0.9807	- 2.3309				-1.4846	1.7662
Vertical Cylinder	1	1.2772	1.1738	1.5951					0.0021	2.1034
Circle	1	1.4776	0.1565	1.5977	0.9339	1.0	3.5917	-4.3788	0.1788	0.4529

Type Joint	Solution No.	Radius R	Location on Fixed Frame X	Y	Location of Joint on Moving Frame D	E	P	α	β	γ
Inverse Circle	1	1.9972	1.5682	1.0119	0.1314	2.1784	-0.7511	1.0000	0.1894	-1.6186

APPENDIX B

COMPUTER PROGRAMS IN FORTRAN IV LANGUAGE FOR THE SPHERE, VERTICAL CYLINDER, CIRCLE, AND INVERSE CIRCLE PROBLEMS

```
CARLSON-D-P*MES.S
      C     PROGRAM FOR SPHERE
            REAL PA,PB,PC,PTH,JC,Q,U,S,A,B,C,AMAT,RR,DPTH,
           1U1,U2,U3,U4,U6,U7,U5
            DIMENSION PA(6),PB(6),PC(6),PTH(6),JC(5),Q(9),R(9),
           1S(9),A(7),B(6),C(6),AMAT(9,9),RR(5),
           2DPTH(6),V(1),RT(17),CO(20),XCAP(5),YCAP(5),ZCAP(5),CKSPH(5)
            INTEGER I,N,MC,NC,NR,KMAX
            N=5
             MC=8
              V(1)=4
            NC=9
             NR=9
             COMPLEX RT,CO
            DO 12 I=1,5
            READ(5,11) PA(I),PB(I),PC(I),DPTH(I)
         11 FORMAT(4F10.5)
            WRITE(6,12) I,PA(I),PB(I),PC(I),DPTH(I)
         12 FORMAT(1H0,5X,5HPOSN ,I1,5X,4HA = ,1PE15.5,5X,4HB = ,1PE15.5,5X,
           14HC = ,1PE15.5,5X,8HTHETA = ,2PE15.5)
            DO 1 I=1,5
         13 PTH(I)=DPTH(I)*(3.1415927/180.0)
            AMAT(I,1)=-2.*PA(I)
            AMAT(I,2)=-2*PB(I)
            AMAT(I,3)=-2.*PC(I)
            AMAT(I,4)=2.*(1.-COS(PTH(I)))
            AMAT(I,5)=2.*SIN(PTH(I))
            AMAT(I,6)=-PA(I)*PA(I)-PB(I)*PB(I)-PC(I)*PC(I)
            AMAT(I,7)=-2.*PA(I)*COS(PTH(I))-2.*PB(I)*SIN(PTH(I))
          1 AMAT(I,8)=-2.*PB(I)*COS(PTH(I))+2.*PA(I)*SIN(PTH(I))
            CALL GJR(AMAT,MC,NR,N,MC,$24,JC,V)
            DO 17 I=1,5
            Q(I+2)=AMAT(I,6)
            R(I+2)=AMAT(I,7)
            S(I+2)=AMAT(I,8)
         17 WRITE(6,3)AMAT(I,6),AMAT(I,7),AMAT(I,8)
          3 FORMAT(1H0,3E25.5)
            A(1)=S(4)
            A(2)=R(4)+S(3)
            A(3)=Q(4)-S(6)
            A(4)=R(3)
            A(5)=Q(3)-R(6)
            A(6)=-Q(6)
            B(1)=R(4)*S(3)-R(3)*S(4)+S(3)*S(3)+S(4)*S(4)
            B(2)=Q(4)*S(3)-Q(3)*S(4)-S(6)*S(3)+S(7)*S(4)
            B(3)=R(3)*S(3)+R(4)*S(4)
            B(4)=Q(3)*S(3)+Q(4)*S(4)-R(6)*S(3)+R(7)*S(4)
            B(5)=-Q(6)*S(3)+Q(7)*S(4)
            C(1)=A(1)*B(3)*B(3)-A(2)*B(1)*B(3)+A(4)*B(1)*B(1)
            C(2)=2.*A(1)*B(3)*B(4)-A(2)*B(1)*B(4)-A(3)*B(1)*B(3)
            C(2)=C(2)+2.*A(4)*B(1)*B(2)+A(5)*B(1)*B(1)-A(2)*B(2)*B(3)
            C(3)=A(1)*B(4)*B(4)+2.*A(1)*B(3)*B(5)-A(2)*B(1)*B(5)-A(3)*B(1)*B(4)
            C(3)=C(3)-A(2)*B(2)*B(4)-A(3)*B(2)*B(3)+A(4)*B(2)*B(2)
            C(3)=C(3)+2.*A(5)*B(1)*B(2)+A(6)*B(1)*B(1)
            C(4)=2.*A(1)*B(4)*B(5)-A(3)*B(1)*B(5)-A(2)*B(2)*B(5)
            C(4)=C(4)-A(3)*B(2)*B(4)+A(5)*B(2)*B(2)+2.*A(6)*B(1)*B(2)
            C(5)=A(1)*B(5)*B(5)-A(3)*B(2)*B(5)+A(6)*B(2)*B(2)
```

```
      DO 8 I=1,5
    8 CO(6-I)=C(I)
      N=4
      EPS=.000001
      KMAX=500
      CALL ROOTCP(CO,N,EPS,KMAX,RI,J,$24)
      IF(J.LT.4)GO TO 24
      DO 9 I=1,4
      RR(I)=REAL(RI(I))
      WRITE(6,9)RI(I)
    9 FORMAT(1H ,15X,2E25.8)
      DO 10 I=1,4
      U1=RR(I)
      U2=-(B(3)*(U1**2)+B(4)*U1+B(5))/(B(1)*U1+B(2))
      U3=Q(3)+R(3)*U1+S(3)*U2
      U4=Q(4)+R(4)*U1+S(4)*U2
      U5=Q(5)+R(5)*U1+S(5)*U2
      U6=Q(6)+R(6)*U1+S(6)*U2
      U7=Q(7)+R(7)*U1+S(7)*U2
      OA=(U1*U1+U2*U2+U3*U3+U4*U4+U5*U5-2.*U6)
      IF(OA.GT.0)GO TO 50
      OA=-OA
      WRITE(6,51) I
   51 FORMAT(1H0,I2,4X,22HOA CHANGED TO POSITIVE)
   50 O=SQRT(OA)
      WRITE(6,20)U1,U2,U3,U4,U5,O
   20 FORMAT(1H0,4HX = ,1PE15.5,2X,4HY = ,1PE15.5,2X,4HD = ,
     11PE15.5,2X,4HE = ,1PE15.5,2X,4HP = ,1PE15.5,2X,4HR = ,
     21PE15.5)
      DO 10 J=1,5
      XCAP(J)=PA(J)+U1*COS(PTH(J))-U2*SIN(PTH(J))
      YCAP(J)=PB(J)+U1*SIN(PTH(J))+U2*COS(PTH(J))
      ZCAP(J)=PC(J)
      CKSPH(J)=SQRT((XCAP(J)-U3)**2+(YCAP(J)-U4)**2+(ZCAP(J)-U5)
     1**2)
   10 WRITE(6,91) CKSPH(J)
   91 FORMAT(1H ,1PE20.5)
   24 WRITE(6,23)
   23 FORMAT(1H0,12HPROGRAM OVER)
   25 WRITE(6,26)
   26 FORMAT(1H0,12HPROGRAM DONE)
      STOP
      END
END PRT#
```

CARLSON-D-P*MEVC.VC

```
C       PROGRAM FOR VERTICAL CYLINDER
        REAL PA,PB,PC,PTH,JC,Q,0,S,A,B,C,AMAT,RR,DPTH,
       1U1,U2,U3,U4,U6,U7
        DIMENSION PA(5),PB(5),PC(5),PTH(5),JC(5),Q(7),R(7),D(4),E(4),F(4),
       1S(7),A(6),B(5),C(5),AMAT(9,9),RR(4),
       2DPTH(5),V(1),RT(17),CO(17)
        INTEGER I,N,MC,NC,NR,KMAX
        N=4
         MC=7
          V(1)=4
        NC=9
         NR=9
         COMPLEX RT,CO
        DO 30 I=1,4
     30 READ(5,11) PA(I),PB(I),PC(I),DPTH(I)
     11 FORMAT(4F10.5)
        DO 31 I=1,4
     31 WRITE(6,12) I,PA(I),PB(I),PC(I),DPTH(I)
     12 FORMAT(1H0,5X,5HPOSN ,I1,5X,4HA = ,1PE15.5,5X,4HB = ,1PE15.5,5X,
       14HC = ,1PE15.5,5X,8HTHETA = ,2PE15.5)
        DO 1 I=1,4
     13 PTH(I)=DPTH(I)*(3.1415927/180.0)
        AMAT(I,1)=-2.*PA(I)
        AMAT(I,2)=-2*PB(I)
        AMAT(I,3)=2.*(1.-COS(PTH(I)))
        AMAT(I,4)=2.*SIN(PTH(I))
        AMAT(I,5)=-PA(I)*PA(I)-PB(I)*PB(I)
        AMAT(I,6)=-2.*PA(I)*COS(PTH(I))-2.*PB(I)*SIN(PTH(I))
      1 AMAT(I,7)=-2.*PB(I)*COS(PTH(I))+2.*PA(I)*SIN(PTH(I))
        CALL GJR(AMAT,NC,NR,N,MC,$24,JC,V)
        DO 6 I=1,4
        D(I)=AMAT(I,5)
        E(I)=AMAT(I,6)
        F(I)=AMAT(I,7)
        WRITE(6,6) D(I),E(I),F(I)
      6 FORMAT(1H ,1P3E20.5)
        Q(3)=D(1)
        Q(4)=D(2)
        Q(6)=D(3)
        Q(7)=D(4)
        R(3)=E(1)
        R(4)=E(2)
        R(6)=E(3)
        R(7)=E(4)
        S(3)=F(1)
        S(4)=F(2)
        S(6)=F(3)
        S(7)=F(4)
        A(1)=S(4)
        A(2)=R(4)+S(3)
        A(3)=Q(4)-S(6)
        A(4)=R(3)
        A(5)=Q(3)-R(6)
        A(6)=-Q(6)
        B(1)=R(4)*S(3)-R(3)*S(4)+S(3)*S(3)+S(4)*S(4)
        B(2)=Q(4)*S(3)-Q(3)*S(4)-S(6)*S(3)+S(7)*S(4)
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      B(3)=R(3)*S(3)+R(4)*S(4)
      B(4)=Q(3)*S(3)+Q(4)*S(4)-R(6)*S(3)+R(7)*S(4)
      B(5)=-Q(6)*S(3)+Q(7)*S(4)
      C(1)=A(1)*B(3)*B(3)-A(2)*B(1)*B(3)+A(4)*B(1)*B(1)
      C(2)=2.*A(1)*B(3)*B(4)-A(2)*B(1)*B(4)-A(3)*B(1)*B(3)
      C(2)=C(2)+2.*A(4)*B(1)*B(2)+A(5)*B(1)*B(1)-A(2)*B(2)*B(3)
      C(3)=A(1)*B(4)*B(4)+2.*A(1)*B(3)*B(5)-A(2)*B(1)*B(5)-A(3)*B(1)*B(4)
      C(3)=C(3)-A(2)*B(2)*B(4)-A(3)*B(2)*B(3)+A(4)*B(2)*B(2)
      C(3)=C(3)+2.*A(5)*B(1)*B(2)+A(6)*B(1)*B(1)
      C(4)=2.*A(1)*B(4)*B(5)-A(3)*B(1)*B(5)-A(2)*B(2)*B(5)
      C(4)=C(4)-A(3)*B(2)*B(4)+A(5)*B(2)*B(2)+2.*A(6)*B(1)*B(2)
      C(5)=A(1)*B(5)*B(5)-A(3)*B(2)*B(5)+A(6)*B(2)*B(2)
      DO 8 I=1,4
      CO(5-I)=C(I+1)
      WRITE(6,8) CO(I)
    8 FORMAT(1H ,5HCO = ,PE20.5)
      N=3
      EPS=.00001
      KMAX=200
      CALL ROOTCP(CO,N,EPS,KMAX,RI,J,$24)
      IF(J.LT.3)GO TO 24
      DO 9 I=1,3
      RR(I)=REAL(RI(I))
      WRITE(6,9)RI(I)
    9 FORMAT(1H ,15X,2E25.8)
      DO 10 I=1,3
      U1=RR(I)
      U2=-(B(3)*(U1**2)+B(4)*U1+B(5))/(B(1)*U1+B(2))
      U3=Q(3)+R(3)*U1+S(3)*U2
      U4=Q(4)+R(4)*U1+S(4)*U2
      U6=Q(6)+R(6)*U1+S(6)*U2
      U7=Q(7)+R(7)*U1+S(7)*U2
      OA=(U1**2+U2**2+U3**2+U4**2-2.*U6)
      IF(OA.GT.0)GO TO 50
      OA=-OA
      WRITE(6,51)I
   51 FORMAT(1H0,I1,22HOA CHANGED TO POSITIVE)
   50 O=SQRT(OA)
      WRITE(6,10)U1,U2,U3,U4,O
   10 FORMAT(1H0,4HX = ,PE18.5,2X,4HY = ,PE18.5,2X,4HU = ,PE18.5,2X,
     14HE = ,PE18.5,2X,4HR = ,PE18.5)
   24 WRITE(6,23)
   23 FORMAT(1H0,12HPROGRAM OVER)
   25 WRITE(6,26)
   26 FORMAT(1H0,12HPROGRAM DONE)
      STOP
      END
END PRT.

@PRT MES.S
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CARLSON-D-P*MEC.C
      C PROGRAM FOR CIRCLE IN THREE DIMENSIONS
            DIMENSION PA(4),PB(4),PC(4),PTH(4),AMAT(7,13),Q(12),
           1R(12),S(12),T(12),V(12),W(12),DPTH(4),V1(2)
            DO 1 I=1,3
          1 READ(5,2) PA(I),PB(I),PC(I),DPTH(I)
          2 FORMAT(4F10.5)
            DO 4 I=1,3
            WRITE(6,3) I,PA(I),PB(I),PC(I),DPTH(I)
          3 FORMAT(1H0,5HPOSN ,I1,5X,4HA = ,1PE15.5,5X,4HB = ,1PE15.5,
           120X,4HC = ,1PE15.5,5X,8HTHETA = ,2PE15.5)
            PTH(I)=DPTH(I)*3.1415927/180.
             AMAT(I,1)=-2.*PC(I)
             AMAT(I,2)=2.*(1.-COS(PTH(I)))
             AMAT(I,3)=2.*SIN(PTH(I))
             AMAT(I,7)=-PA(I)*PA(I)-PB(I)*PB(I)-PC(I)*PC(I)
             AMAT(I,8)=-2.*PA(I)*COS(PTH(I))-2.*PB(I)*SIN(PTH(I))
             AMAT(I,9)=-2.*PB(I)*COS(PTH(I))+2.*PA(I)*SIN(PTH(I))
             AMAT(I,10)=2.*PA(I)
             AMAT(I,11)=2.*PB(I)
             AMAT(I+3,4)=-PC(I)
             AMAT(I+3,5)=-SIN(PTH(I))
             AMAT(I+3,6)=(1.-COS(PTH(I)))
             AMAT(I+3,7)=PA(I)
          4 AMAT(I+3,12)=PB(I)
             N=6
             MC=12
             V1(1)=4
             NC=13
             NR=7
             CALL GJR(AMAT,NC,NR,N,MC,$300,JC,V1)
             DO 6 I=1,3
             Q(I+4)=AMAT(I,7)
             R(I+4)=AMAT(I,8)
             S(I+4)=AMAT(I,9)
             T(I+4)=AMAT(I,10)
             V(I+4)=AMAT(I,11)
             Q(I+8)=AMAT(I+3,7)
             W(I+8)=AMAT(I+3,12)
             WRITE(6,5) Q(I+4),R(I+4),S(I+4),T(I+4)
          5 FORMAT(1H ,4E15.5)
          6 WRITE(6,7) V(I+4),Q(I+8),W(I+8)
          7 FORMAT(1H ,5X,3E15.5)
            DIMENSION A(20),B(20),C(20),D(21),E(20),F(12),
           1RRF(8),FC(11),UA(16),UB(17),
           2P5(17),ARGU2(9)
             A(1)=1.
             A(2)=1.
             A(3)=V(7)-T(6)
             A(4)=-V(6)-T(7)
             A(5)=V(6)*T(7)-V(7)*T(6)
             A(6)=R(7)
             A(7)=S(7)-R(6)
             A(8)=-S(6)
             A(9)=Q(7)+R(6)*T(7)-R(7)*T(6)
             A(10)=-Q(6)+S(6)*T(7)-S(7)*T(6)
             A(11)=T(7)*Q(6)-T(6)*Q(7)
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      B(1)=-W(9)*T(5)
      B(2)=Q(9)*T(5)+1.
      B(3)=Q(11)*W(9)*T(5)-Q(9)*W(11)*T(5)-W(11)
      B(4)=-W(9)*R(5)
      B(5)=Q(9)*S(5)
      B(6)=Q(9)*R(5)-W(9)*S(5)
      B(7)=-W(9)*V(5)-1.
      B(8)=Q(9)*V(5)
      B(9)=Q(11)*W(9)*R(5)-Q(9)*W(11)*R(5)-W(9)*Q(5)+W(11)
      B(10)=Q(11)*W(9)*S(5)-Q(9)*W(11)*S(5)+Q(9)*Q(5)-Q(11)
      B(11)=Q(11)*W(9)*V(5)-Q(9)*W(11)*V(5)+Q(11)
      B(12)=Q(11)*W(9)*Q(5)-Q(9)*W(11)*Q(5)
      C(1)=B(7)
      C(2)=-B(1)+B(8)
      C(3)=-B(2)
      C(4)=B(11)-B(7)*T(6)+B(1)*V(6)
      C(5)=-B(3)-B(8)*T(6)+B(2)*V(6)
      C(6)=B(3)*V(6)-B(11)*T(6)
      C(7)=B(4)
      C(8)=B(6)
      C(9)=B(5)
      C(10)=B(9)-B(4)*T(6)+B(1)*R(6)
      C(11)=B(10)-B(6)*T(6)+B(1)*S(6)+B(2)*R(6)
      C(12)=B(2)*S(6)-B(5)*T(6)
      C(13)=B(12)-B(9)*T(6)+B(1)*Q(6)+B(3)*R(6)
      C(14)=-B(10)*T(6)+B(2)*Q(6)+B(3)*S(6)
      C(15)=-B(12)*T(6)+B(3)*Q(6)
      D(1)=A(1)*C(7)
      D(2)=A(1)*C(8)
      D(3)=A(1)*C(9)+A(2)*C(7)
      D(4)=A(2)*C(8)
      D(5)=A(2)*C(9)
      D(6)=A(1)*C(10)+A(3)*C(7)-A(6)*C(1)
      D(7)=A(1)*C(11)+A(3)*C(8)+A(4)*C(7)-A(6)*C(2)-A(7)*C(1)
      D(8)=A(1)*C(12)+A(2)*C(10)+A(3)*C(9)+A(4)*C(8)-A(6)*C(3)
     1-A(7)*C(2)-A(8)*C(1)
      D(9)=A(2)*C(11)+A(4)*C(9)-A(7)*C(3)-A(8)*C(2)
      D(10)=A(2)*C(12)-A(8)*C(3)
      D(11)=A(1)*C(13)+A(3)*C(10)+A(5)*C(7)-A(6)*C(4)-A(9)*C(1)
      D(12)=A(1)*C(14)+A(3)*C(11)+A(4)*C(10)+A(5)*C(8)-A(6)*C(5)
     1-A(7)*C(4)-A(9)*C(2)-A(10)*C(1)
      D(13)=A(2)*C(13)+A(3)*C(12)+A(4)*C(11)+A(5)*C(9)-A(7)*C(5)
     1-A(8)*C(4)-A(9)*C(3)-A(10)*C(2)
      D(14)=A(2)*C(14)+A(4)*C(12)-A(8)*C(5)-A(10)*C(3)
      D(15)=A(1)*C(15)+A(3)*C(13)+A(5)*C(10)-A(6)*C(6)-A(9)*C(4)
     1-A(11)*C(1)
      D(16)=A(3)*C(14)+A(4)*C(13)+A(5)*C(11)-A(7)*C(6)-A(9)*C(5)
     1-A(10)*C(4)-A(11)*C(2)
      D(17)=A(2)*C(15)+A(4)*C(14)+A(5)*C(12)-A(8)*C(6)-A(10)*C(5)
     1-A(11)*C(3)
      D(18)=A(3)*C(15)+A(5)*C(13)-A(9)*C(6)-A(11)*C(4)
      D(19)=A(4)*C(15)+A(5)*C(14)-A(10)*C(6)-A(11)*C(5)
      D(20)=A(5)*C(15)-A(11)*C(6)
      X=(W(11)-Q(10))/2.
      Y=Q(11)+W(10)
      Z=((Q(10)+W(11))**2)/4.-Q(11)*W(10)
     WRITE(6,24) X,Y,Z
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```
   24 FORMAT(1H0,3E15.5,/)
        E(1)=D(1)-D(3)+D(5)
        E(2)=D(6)+D(2)*X+D(3)*Y-D(8)-3.*D(4)*X+D(10)-2.*D(5)*Y
        E(3)=D(11)+X*D(7)+X*X*D(3)+Z*D(3)+Y*D(8)-D(13)+3.*X*Y*D(4)
      1-3.*X*D(9)-6.*X*X*D(5)+Y*Y*D(5)-2.*Z*D(5)-2.*Y*D(10)
        E41=D(15)+X*D(12)+X*X*D(8)+Z*D(8)+Y*D(13)-D(17)
      1+X*X*X*D(4)+3.*X*Z*D(4)+3.*X*Y*D(9)
        E42=-3.*X*D(14)+6.*X*X*Y*D(5)+2.*Y*Z*D(5)-6.*X*X*D(10)+
      1Y*Y*D(10)-2.*Z*D(10)
        E(4)=E41+E42
        E51=D(18)+X*D(16)+X*X*D(13)+Z*D(13)+Y*D(17)+X*X*X*D(9)+3.*X
      1*Z*D(9)
        E52=3.*X*Y*D(14)+X**4*D(5)+6.*X*X*Z*D(5)+Z*Z*D(5)+6.*X*X*Y
      1*D(10)+2.*Y*Z*D(10)
        E(5)=E51+E52
        E(6)=D(20)+X*D(19)+X*X*D(17)+Z*D(17)+X**3*D(14)+3.*X*Z*
      1D(14)+X**4*D(10)+6.*X*X*Z*D(10)+Z*Z*D(10)
        E(7)=D(2)-D(4)
        E(8)=D(7)+2.*X*D(3)+Y*D(4)-D(9)-4.*X*D(5)
        E(9)=D(12)+2.*X*D(8)-D(14)+Y*D(9)+3.*X*X*D(4)+Z*D(4)-4.*
      1X*D(10)+4.*X*Y*D(5)
        E(10)=D(16)+2.*X*D(13)+3.*X*Y*D(9)+Z*D(9)+Y*D(14)+4.*X*X*
      1X*D(5)+4.*X*Z*D(5)+4.*X*Y*D(10)
        E(11)=D(19)+2.*X*D(17)+3.*X*X*D(14)+Z*D(14)+4.*X**3*D(10)
      1+4.*X*Z*D(10)
        F(1)=E(2)*E(2)+E(8)*E(8)
        F(2)=2.*E(2)*E(3)+2.*E(9)*E(8)-Y*E(8)*E(8)
        F(3)=E(3)*E(3)+2.*E(2)*E(4)+E(9)*E(9)+2.*E(8)*E(10)-2.*Y*
      1E(8)*E(9)-Z*E(8)*E(8)
        F(4)=2.*E(2)*E(5)+2.*E(4)*E(3)+2.*E(8)*E(11)+2.*E(9)*E(10)
      1-Y*E(9)*E(9)-2.*Y*E(8)*E(10)-2.*Z*E(8)*E(9)
        F(5)=E(4)*E(4)+2.*E(2)*E(6)+2.*E(3)*E(5)+E(10)*E(10)
      1+2.*E(9)*E(11)-2.*Y*E(8)*E(11)-2.*Y*E(9)*E(
      210)-Z*E(9)*E(9)-2.*Z*E(8)*E(10)
       F(6)=2.*E(3)*E(6)+2.*E(4)*E(5)+2.*E(10)*E(11)-Y*
      1E(10)*E(10)-2.*Y*E(9)*E(11)-2.*Z*E(8)*E(11)-2.*
      2Z*E(9)*E(10)
       F(7)=E(5)*E(5)+2.*E(4)*E(6)+E(11)*E(11)-2.*Y*E(10
      1)*E(11)-Z*E(10)*E(10)-2.*Z*E(9)*E(11)
       F(8)=2.*E(5)*E(6)-Y*E(11)*E(11)-2.*Z*E(10)*E(11)
       F(9)=E(6)*E(6)-Z*E(11)*E(11)
       DO 26 J=1,20
   26 WRITE(6,27) A(J),B(J),C(J),D(J),E(J)
   27 FORMAT(1H ,4E16.6,3X,1E16.6)
       COMPLEX FC
       DO 8 I=1,9
       FC(10-I)=F(I)
    8 WRITE(6,11) F(I)
   11 FORMAT(1H ,1PE35.10)
        N=8
        EPS=.0000001
        KMAX=300
       DIMENSION RF(9)
       COMPLEX RF
       CALL ROOTCP(FC,N,EPS,KMAX,RF,J,$300)
       IF(J.LT.8)GO TO 300
       DO 9 I=1,8
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      RRF(I)=REAL(RF(I))
    9 WRITE(6,10)RF(I)
   10 FORMAT(1H ,10X,1P2E15.5)
      DO 12 I=1,8
   12 UA(I)=RRF(I)
      DO 15 L=1,5,4
      M1=L+3
      DO 14 J=L,M1
      ARGU2(J)=-UA(J)*UA(J)+Y*UA(J)+Z
      WRITE(6,31) ARGU2(J)
   31 FORMAT(1H0,1PE20.5)
      IF(ARGU2(J).GT.0)GO TO 28
      ARGU2(J)=ABS(ARGU2(J))
      WRITE(6,29) J
   29 FORMAT(1H0,4HJ = ,I2,3X,25HARGU2 CHANGED TO POSITIVE)
   28 UB(J*2-1)=X+SQRT(ARGU2(J))
   14 UB(J*2)=X-SQRT(ARGU2(J))
      M2=(2*L)-1
      M3=(2*L)+6
      DO 15 K=M2,M3
      K1=(K+1)/2
      U1=UA(K1)
      U2=UB(K)
      P51=((((D(1)*U1+D(2)*U2)*U1+D(3)*U2**2)*U1+D(4)*
     1U2**3)*U1+D(5)*U2**4)*U1
      P52=(((D(6)*U1+D(7)*U2)*U1+D(8)*U2**2)*U1+D(9)*
     1U2**3)*U1+D(10)*U2**4
      P53=((D(11)*U1+D(12)*U2)*U1+D(13)*U2**2)*U1+D(14)*U2**3
      P54=(D(15)*U1+D(16)*U2)*U1+D(17)*U2**2+D(18)*U1+D(19)*
     1U2+D(20)
      P5(K)=P51+P52+P53+P54
   15 WRITE(6,16) UA(K),UB(K),P5(K)
   16 FORMAT(1H ,1P3E15.5)
      DIMENSION XCAP(4),YCAP(4),ZCAP(4),CKSPH(4),CKPLN(4)
      DO 17 I=1,16
      K2=(I+1)/2
      U1=UA(K2)
      U2=UB(I)
      ARGU3=ARGU2(K2)
      U4B=-(C(7)*U1**3+C(8)*U1*U1*U2+C(9)*U1*U2*U2+C(10)
     1*U1*U1+C(11)*U1*U2+C(12)*U2*U2+C(13)*U1+C(14)*U2+C(15)
     2)/(C(1)*U1*U1+C(2)*U1*U2+C(3)*U2*U2+C(4)*U1+C(5)*U2
     3+C(6))
      U4=U4B
      U3A=(-U2*U4+R(6)*U1+S(6)*U2+V(6)*U4+Q(6))/(U1-
     1T(6))
      U3B=(U1*U4+R(7)*U1+S(7)*U2+V(7)*U4+Q(7))/(U2-
     1T(7))
      U3C=-(B(4)*U1*U1+B(5)*U2*U2+B(6)*U1*U2+B(7)*U1*U4+
     1B(8)*U2*U4+B(9)*U1+B(10)*U2+B(11)*U4+B(12))/(B(1)*
     2U1+B(2)*U2+B(3))
      U3=U3A
      U8A=(U2+Q(10))/(U1-W(10))
      U8B=(-U1+Q(11))/(U2-W(11))
      U8C=-(Q(9)*R(5)*U1+Q(9)*S(5)*U2+(Q(9)*T(5)+1.)*
     1U3+Q(9)*V(5)*U4+(Q(9)*Q(5)-Q(11)))/(W(9)*R(5)*U1
     2+W(9)*S(5)*U2+W(9)*T(5)*U3+(W(9)*V(5)+1.)*U4+
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     3(W(9)*Q(5)-W(11)))
      U8=U8A
      U5=Q(5)+R(5)*U1+S(5)*U2+T(5)*U3+V(5)*U4
      U6=Q(6)+R(6)*U1+S(6)*U2+T(6)*U3+V(6)*U4
      U7=Q(7)+R(7)*U1+S(7)*U2+T(7)*U3+V(7)*U4
      U9=Q(9)+W(9)*U8
     BETA=U8
     GAMMA=U9
     ALPHA=1.0
      U10=Q(10)+W(10)*U8
      U11=Q(11)+W(11)*U8
      RA=SQRT(U1*U1+U2*U2+U3*U3+U4*U4+U5*U5-2.*U6)
      WRITE(6,18)U1,U2,U3,U4,U5,ALPHA,BETA,GAMMA,RA
   18 FORMAT(1H0,4HX = ,1PE15.5,7X,4HY = ,1PE15.5,8X,
     14HD = ,1PE15.5,5X,4HE = ,1PE15.5,/,1H ,4HP = ,1PE15.5,
     25X,8HALPHA = ,1PE15.5,5X,7HBETA = ,1PE15.5,5X,8HGAMMA = ,1PE15.5
     3,5X,4HR = ,PE15.5)
      DO 19 J=1,3
      XCAP(J)=PA(J)+U1*COS(PTH(J))-U2*SIN(PTH(J))
      YCAP(J)=PB(J)+U1*SIN(PTH(J))+U2*COS(PTH(J))
      ZCAP(J)=PC(J)
      CKSPH(J)=SQRT((XCAP(J)-U3)**2+(YCAP(J)-U4)**2+(ZCAP(J)
     1-U5)**2)
      CKPLN(J)=(XCAP(J)-U3)+U8*(YCAP(J)-U4)+U9*(ZCAP(J)-U5)
   19 WRITE(6,20) CKSPH(J),CKPLN(J)
   20 FORMAT(1H ,1P2E15.5)
      CHK9=U1*U3+U2*U4-U6
      CHK10=U2*U3-U1*U4-U7
      CHK11=U1*U8-U2-U10
      CHK12=U2*U8+U1-U11
      CHK13=U4*U8+U5*U9+U3-U11
      WRITE(6,21)CHK9,CHK10,CHK11,CHK12,CHK13
   17 WRITE(6,22)U3A,U3B,U3C,U4B,U8A,U8B,U8C
   21 FORMAT(1H ,1P4E15.5,1P2E15.5)
   22 FORMAT(1H ,1P4E15.5,1P3E15.5)
  300 WRITE(6,23)
   23 FORMAT(1H ,12HPROGRAM OVER)
      STOP
      END
END PRT.

@PRT MEIC.C
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```
CARLSON-D-P*MEIC.C
      C PROGRAM FOR CIRCLE IN THREE DIMENSIONS
            DIMENSION PA(4),PB(4),PC(4),PTH(4),AMAT(7,13),Q(12),
           1R(12),S(12),T(12),V(12),W(12),DPTH(4),V1(2)
            DO 1 I=1,3
          1 READ(5,2) PA(I),PB(I),PC(I),DPTH(I)
          2 FORMAT(4F10.5)
            DO 4 I=1,3
            WRITE(6,3) I,PA(I),PB(I),PC(I),DPTH(I)
          3 FORMAT(1H0,5HPOSN ,I1,5X,4HA = ,1PE15.5,5X,4HB = ,1PE15.5,
           120X,4HC = ,1PE15.5,5X,8HTHETA = ,2PE15.5)
            PTH(I)=DPTH(I)*3.1415927/180.
            TPAI=-PA(I)*COS(PTH(I))-PB(I)*SIN(PTH(I))
            PB(I)=PA(I)*SIN(PTH(I))-PB(I)*COS(PTH(I))
            PC(I)=-PC(I)
            PTH(I)=-PTH(I)
            PA(I)=TPAI
              AMAT(I,1)=-2.*PC(I)
              AMAT(I,2)=2.*(1.-COS(PTH(I)))
              AMAT(I,3)=2.*SIN(PTH(I))
              AMAT(I,7)=-PA(I)*PA(I)-PB(I)*PB(I)-PC(I)*PC(I)
              AMAT(I,8)=-2.*PA(I)*COS(PTH(I))-2.*PB(I)*SIN(PTH(I))
              AMAT(I,9)=-2.*PB(I)*COS(PTH(I))+2.*PA(I)*SIN(PTH(I))
              AMAT(I,10)=2.*PA(I)
              AMAT(I,11)=2.*PB(I)
              AMAT(I+3,4)=-PC(I)
              AMAT(I+3,5)=-SIN(PTH(I))
              AMAT(I+3,6)=(1.-COS(PTH(I)))
              AMAT(I+3,7)=PA(I)
          4 AMAT(I+3,12)=PB(I)
              N=6
              MC=12
              V1(1)=4
              NC=13
              NR=7
              CALL GJR(AMAT,NC,NR,N,MC,$300,JC,V1)
              DO 6 I=1,3
              Q(I+4)=AMAT(I,7)
              R(I+4)=AMAT(I,8)
              S(I+4)=AMAT(I,9)
              T(I+4)=AMAT(I,10)
              V(I+4)=AMAT(I,11)
              Q(I+8)=AMAT(I+3,7)
              W(I+8)=AMAT(I+3,12)
              WRITE(6,5) Q(I+4),R(I+4),S(I+4),T(I+4)
          5 FORMAT(1H ,4E15.5)
          6 WRITE(6,7) V(I+4),Q(I+8),W(I+8)
          7 FORMAT(1H ,5X,3E15.5)
            DIMENSION A(20),B(20),C(20),D(21),E(20),F(12),
           1RRF(8),FC(11),UA(16),UB(17),
           2P5(17),ARGU2(9)
              A(1)=1.
              A(2)=1.
              A(3)=V(7)-T(6)
              A(4)=-V(6)-T(7)
              A(5)=V(6)*T(7)-V(7)*T(6)
              A(6)=R(7)
```

```
      A(7)=S(7)-R(6)
      A(8)=-S(6)
      A(9)=Q(7)+R(6)*T(7)-R(7)*T(6)
      A(10)=-Q(6)+S(6)*T(7)-S(7)*T(6)
      A(11)=T(7)*Q(6)-T(6)*Q(7)
      B(1)=-W(9)*T(5)
      B(2)=Q(9)*T(5)+1.
      B(3)=Q(11)*W(9)*T(5)-Q(9)*W(11)*T(5)-W(11)
      B(4)=-W(9)*R(5)
      B(5)=Q(9)*S(5)
      B(6)=Q(9)*R(5)-W(9)*S(5)
      B(7)=-W(9)*V(5)-1.
      B(8)=Q(9)*V(5)
      B(9)=Q(11)*W(9)*R(5)-Q(9)*W(11)*R(5)-W(9)*Q(5)+W(11)
      B(10)=Q(11)*W(9)*S(5)-Q(9)*W(11)*S(5)+Q(9)*Q(5)-Q(11)
      B(11)=Q(11)*W(9)*V(5)-Q(9)*W(11)*V(5)+Q(11)
      B(12)=Q(11)*W(9)*Q(5)-Q(9)*W(11)*Q(5)
      C(1)=B(7)
      C(2)=-B(1)+B(8)
      C(3)=-B(2)
      C(4)=B(11)-B(7)*T(6)+B(1)*V(6)
      C(5)=-B(3)-B(8)*T(6)+B(2)*V(6)
      C(6)=B(3)*V(6)-B(11)*T(6)
      C(7)=B(4)
      C(8)=B(6)
      C(9)=B(5)
      C(10)=B(9)-B(4)*T(6)+B(1)*R(6)
      C(11)=B(10)-B(6)*T(6)+B(1)*S(6)+B(2)*R(6)
      C(12)=B(2)*S(6)-B(5)*T(6)
      C(13)=B(12)-B(9)*T(6)+B(1)*Q(6)+B(3)*R(6)
      C(14)=-B(10)*T(6)+B(2)*Q(6)+B(3)*S(6)
      C(15)=-B(12)*T(6)+B(3)*Q(6)
      D(1)=A(1)*C(7)
      D(2)=A(1)*C(8)
      D(3)=A(1)*C(9)+A(2)*C(7)
      D(4)=A(2)*C(8)
      D(5)=A(2)*C(9)
      D(6)=A(1)*C(10)+A(3)*C(7)-A(6)*C(1)
      D(7)=A(1)*C(11)+A(3)*C(8)+A(4)*C(7)-A(6)*C(2)-A(7)*C(1)
      D(8)=A(1)*C(12)+A(2)*C(10)+A(3)*C(9)+A(4)*C(8)-A(6)*C(3)
     1-A(7)*C(2)-A(8)*C(1)
      D(9)=A(2)*C(11)+A(4)*C(9)-A(7)*C(3)-A(8)*C(2)
      D(10)=A(2)*C(12)-A(8)*C(3)
      D(11)=A(1)*C(13)+A(3)*C(10)+A(5)*C(7)-A(6)*C(4)-A(9)*C(1)
      D(12)=A(1)*C(14)+A(3)*C(11)+A(4)*C(10)+A(5)*C(8)-A(6)*C(5)
     1-A(7)*C(4)-A(9)*C(2)-A(10)*C(1)
      D(13)=A(2)*C(13)+A(3)*C(12)+A(4)*C(11)+A(5)*C(9)-A(7)*C(5)
     1-A(8)*C(4)-A(9)*C(3)-A(10)*C(2)
      D(14)=A(2)*C(14)+A(4)*C(12)-A(8)*C(5)-A(10)*C(3)
      D(15)=A(1)*C(15)+A(3)*C(13)+A(5)*C(10)-A(6)*C(6)-A(9)*C(4)
     1-A(11)*C(1)
      D(16)=A(3)*C(14)+A(4)*C(13)+A(5)*C(11)-A(7)*C(6)-A(9)*C(5)
     1-A(10)*C(4)-A(11)*C(2)
      D(17)=A(2)*C(15)+A(4)*C(14)+A(5)*C(12)-A(8)*C(6)-A(10)*C(5)
     1-A(11)*C(3)
      D(18)=A(3)*C(15)+A(5)*C(13)-A(9)*C(6)-A(11)*C(4)
      D(19)=A(4)*C(15)+A(5)*C(14)-A(10)*C(6)-A(11)*C(5)
```

```
      D(20)=A(5)*C(15)-A(11)*C(6)
      X=(W(11)-Q(10))/2.
      Y=Q(11)+W(10)
      Z=((Q(10)+W(11))**2)/4.-Q(11)*W(10)
     WRITE(6,24) X,Y,Z
  24 FORMAT(1H0,3E15.5,/)
      E(1)=D(1)-D(3)+D(5)
      E(2)=D(6)+D(2)*X+D(3)*Y-D(8)-3.*D(4)+X+D(10)-2.*D(5)*Y
      E(3)=D(11)+X*D(7)+X*X*D(3)+Z*D(3)+Y*D(8)-D(13)+3.*X*Y*D(4)
     1-3.*X*D(9)-6.*X*X*D(5)+Y*Y*D(5)-2.*Z*D(5)-2.*Y*D(10)
      E41=D(15)+X*D(12)+X*X*D(8)+Z*D(8)+Y*D(13)-D(17)
     1+X*X*X*D(4)+3.*X*Z*D(4)+3.*X*Y*D(9)
      E42=-3.*X*D(14)+6.*X*X*Y*D(5)+2.*Y*Z*D(5)-6.*X*X*D(10)+
     1Y*Y*D(10)-2.*Z*D(10)
      E(4)=E41+E42
      E51=D(18)+X*D(16)+X*X*D(13)+Z*D(13)+Y*D(17)+X*X*X*D(9)+3.*X
     1*Z*D(9)
      E52=3.*X*Y*D(14)+X**4*D(5)+6.*X*X*Z*D(5)+Z*Z*D(5)+6.*X*X*Y
     1*D(10)+2.*Y*Z*D(10)
      E(5)=E51+E52
      E(6)=D(20)+X*D(19)+X*X*D(17)+Z*D(17)+X**3*D(14)+3.*X*Z*
     1D(14)+X**4*D(10)+6.*X*X*Z*D(10)+Z*Z*D(10)
      E(7)=D(2)-D(4)
      E(8)=D(7)+2.*X*D(3)+Y*D(4)-D(9)-4.*X*D(5)
      E(9)=D(12)+2.*X*D(8)-D(14)+Y*D(9)+3.*X*X*D(4)+Z*D(4)-4.*
     1X*D(10)+4.*X*Y*D(5)
      E(10)=D(16)+2.*X*D(13)+3.*X*X*D(9)+Z*D(9)+Y*D(14)+4.*X*X*
     1X*D(5)+4.*X*Z*D(5)+4.*X*Y*D(10)
      E(11)=D(19)+2.*X*D(17)+3.*X*X*D(14)+Z*D(14)+4.*X**3*D(10)
     1+4.*X*Z*D(10)
      F(1)=E(2)*E(2)+E(8)*E(8)
      F(2)=2.*E(2)*E(3)+2.*E(9)*E(8)-Y*E(8)*E(8)
      F(3)=E(3)*E(3)+2.*E(2)*E(4)+E(9)*E(9)+2.*E(8)*E(10)-2.*Y*
     1E(8)*E(9)-Z*E(8)*E(8)
      F(4)=2.*E(2)*E(5)+2.*E(4)*E(3)+2.*E(8)*E(11)+2.*E(9)*E(10)
     1-Y*E(9)*E(9)-2.*Y*E(8)*E(10)-2.*Z*E(8)*E(9)
      F(5)=E(4)*E(4)+2.*E(2)*E(6)+2.*E(3)*E(5)+E(10)*E(10)
     1+2.*E(9)*E(11)-2.*Y*E(8)*E(11)-2.*Y*E(9)*E(
     210)-Z*E(9)*E(9)-2.*Z*E(8)*E(10)
     F(6)=2.*E(3)*E(6)+2.*E(4)*E(5)+2.*E(10)*E(11)-Y*
     1E(10)*E(10)-2.*Y*E(9)*E(11)-2.*Z*E(8)*E(11)-2.*
     2Z*E(9)*E(10)
     F(7)=E(5)*E(5)+2.*E(4)*E(6)+E(11)*E(11)-2.*Y*E(10
     1)*E(11)-Z*E(10)*E(10)-2.*Z*E(9)*E(11)
     F(8)=2.*E(5)*E(6)-Y*E(11)*E(11)-2.*Z*E(10)*E(11)
     F(9)=E(6)*E(6)-Z*E(11)*E(11)
     DO 26 J=1,20
  26 WRITE(6,27) A(J),B(J),C(J),D(J),E(J)
  27 FORMAT(1H ,4E16.6,3X,E16.6)
     COMPLEX FC
     DO 8 I=1,9
     FC(10-I)=F(I)
   8 WRITE(6,11) F(I)
  11 FORMAT(1H ,1PE35.10)
      N=8
      EPS=.0000001
      KMAX=300
```

```
      DIMENSION RF(9)
      COMPLEX RF
      CALL ROOTCP(FC,N,EPS,KMAX,RF,J,$300)
      IF(J.LT.8)GO TO 300
      DO 9 I=1,8
       RRF(I)=REAL(RF(I))
    9 WRITE(6,10)RF(I)
   10 FORMAT(1H ,10X,1P2E15.5)
       DO 12 I=1,8
   12 UA(I)=RRF(I)
      DO 15 L=1,5,4
       M1=L+3
       DO 14 J=L,M1
       ARGU2(J)=-UA(J)*UA(J)+Y*UA(J)+Z
      WRITE(6,31) ARGU2(J)
   31 FORMAT(1H0,1PE20.5)
      IF(ARGU2(J).GT.0)GO TO 28
       ARGU2(J)=ABS(ARGU2(J))
       WRITE(6,29) J
   29 FORMAT(1H0,4HJ = ,I2,3X,25HARGU2 CHANGED TO POSITIVE)
   28 UB(J*2-1)=X+SQRT(ARGU2(J))
   14 UB(J*2)=X-SQRT(ARGU2(J))
       M2=(2*L)-1
       M3=(2*L)+6
       DO 15 K=M2,M3
      K1=(K+1)/2
       U1=UA(K1)
       U2=UB(K)
       P51=((((D(1)*U1+D(2)*U2)*U1+D(3)*U2**2)*U1+D(4)*
     1U2**3)*U1+D(5)*U2**4)*U1
       P52=(((D(6)*U1+D(7)*U2)*U1+D(8)*U2**2)*U1+D(9)*
     1U2**3)*U1+D(10)*U2**4
       P53=((D(11)*U1+D(12)*U2)*U1+D(13)*U2**2)*U1+D(14)*U2**3
       P54=(D(15)*U1+D(16)*U2)*U1+D(17)*U2**2+D(18)*U1+D(19)*
     1U2+D(20)
       P5(K)=P51+P52+P53+P54
   15 WRITE(6,16) UA(K),UB(K),P5(K)
   16 FORMAT(1H ,1P3E15.5)
      DIMENSION XCAP(4),YCAP(4),ZCAP(4),CKSPH(4),CKPLN(4)
       DO 17 I=1,16
      K2=(I+1)/2
       U1=UA(K2)
       U2=UB(I)
       ARGU3=ARGU2(K2)
       U4B=-(C(7)*U1**3+C(8)*U1*U1*U2+C(9)*U1*U2*U2+C(10)
     1*U1*U1+C(11)*U1*U2+C(12)*U2*U2+C(13)*U1+C(14)*U2+C(15)
     2)/(C(1)*U1*U1+C(2)*U1*U2+C(3)*U2*U2+C(4)*U1+C(5)*U2
     3+C(6))
       U4=U4B
       U3A=(-U2*U4+R(6)*U1+S(6)*U2+V(6)*U4+Q(6))/(U1-
     1T(6))
       U3B=(U1*U4+R(7)*U1+S(7)*U2+V(7)*U4+Q(7))/(U2-
     1T(7))
       U3C=-(B(4)*U1*U1+B(5)*U2*U2+B(6)*U1*U2+B(7)*U1*U4+
     1B(8)*U2*U4+B(9)*U1+B(10)*U2+B(11)*U4+B(12))/(B(1)*
     2U1+B(2)*U2+B(3))
       U3=U3A
```

```
      U8A=(U2+Q(10))/(U1-W(10))
      U8B=(-U1+Q(11))/(U2-W(11))
      U8C=-(Q(9)*R(5)*U1+Q(9)*S(5)*U2+(Q(9)*T(5)+1.)*
     1U3+Q(9)*V(5)*U4+(Q(9)*Q(5)-Q(11)))/(W(9)*R(5)*U1
     2+W(9)*S(5)*U2+W(9)*T(5)*U3+(W(9)*V(5)+1.)*U4+
     3(W(9)*Q(5)-W(11)))
      U8=U8A
      U5=Q(5)+R(5)*U1+S(5)*U2+T(5)*U3+V(5)*U4
      U6=Q(6)+R(6)*U1+S(6)*U2+T(6)*U3+V(6)*U4
      U7=Q(7)+R(7)*U1+S(7)*U2+T(7)*U3+V(7)*U4
      U9=Q(9)+W(9)*U8
     BETA=U8
     GAMMA=U9
     ALPHA=1.0
      U10=Q(10)+W(10)*U8
      U11=Q(11)+W(11)*U8
      RA=SQRT(U1*U1+U2*U2+U3*U3+U4*U4+U5*U5-2.*U6)
      WRITE(6,18)U1,U2,U3,U4,U5,ALPHA,BETA,GAMMA,RA
   18 FORMAT(1H0,4HX = ,1PE15.5,7X,4HY = ,1PE15.5,8X,
     14HD = ,1PE15.5,5X,4HE = ,1PE15.5,/,1H ,4HP = ,1PE15.5,
     25X,8HALPHA = ,1PE15.5,5X,7HBETA = ,1PE15.5,5X,8HGAMMA = ,1PE15.5
     3,5X,4HR = ,1PE15.5)
      DO 19 J=1,3
      XCAP(J)=PA(J)+U1*COS(PTH(J))-U2*SIN(PTH(J))
      YCAP(J)=PB(J)+U1*SIN(PTH(J))+U2*COS(PTH(J))
      ZCAP(J)=PC(J)
      CKSPH(J)=SQRT((XCAP(J)-U3)**2+(YCAP(J)-U4)**2+(ZCAP(J)
     1-U5)**2)
      CKPLN(J)=(XCAP(J)-U3)+U6*(YCAP(J)-U4)+U9*(ZCAP(J)-U5)
   19 WRITE(6,20) CKSPH(J),CKPLN(J)
   20 FORMAT(1H ,1P2E15.5)
      CHK9=U1*U3+U2*U4-U6
      CHK10=U2*U3-U1*U4-U7
      CHK11=U1*U8-U2-U10
      CHK12=U2*U8+U1-U11
      CHK13=U4*U8+U5*U9+U3-U11
      WRITE(6,21)CHK9,CHK10,CHK11,CHK12,CHK13
   17 WRITE(6,22)U3A,U3B,U3C,U4B,U8A,U8B,U8C
   21 FORMAT(1H ,1P4E15.5,1P2E15.5)
   22 FORMAT(1H ,1P4E15.5,1P3E15.5)
  300 WRITE(6,23)
   23 FORMAT(1H ,12HPROGRAM OVER)
      STOP
      END
END PRT*
```

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